A nonlinear relaxation formulation of the $p$-curl problem modelling high-temperature superconductors: a modified Yee’s scheme

Y.-M. Law and M. Laforest

Département de mathématiques et de génie industriel, École Polytechnique de Montréal, Montréal, C.P. 6079, succ. centre-ville, Québec, Canada, H3C 3A7, ID (yann-meing.law-kam-cio@polymtl.ca, marc.laforest@polymtl.ca)

Abstract

We propose a new tool for the study of the parabolic $p$-curl problem modelling the magnetic field in a high-temperature superconductor (HTS), a problem involving an operator $\nabla \times (|\nabla \times \textbf{H}|^{p-2} \nabla \times \textbf{H})$ analogous to the $p$-Laplacian. This so-called nonlinear $p$-curl problem is computationally expensive in three space dimensions because sharp fronts in the magnetic field develop near the surface of HTS. The proposed technique is a relaxation model of the $p$-curl problem which is shown to provide a consistent and stable approximation according to first, an inner/outer layer analysis and secondly, according to a time scale expansion. The relaxation model leads to monotone approximations of the front because of the presence of anisotropic diffusion. Most importantly, it can be used to develop new schemes that inherit the properties of the relaxation model.

In order to demonstrate the relevance of the relaxation model, this paper applies the approximation to the construction of a second-order nonlinear finite-difference time-domain (FDTD) method, similar to Yee’s scheme. It is shown that the natural discretization of the $p$-curl inspired by Yee’s scheme leads to an unstable scheme, while a discretization based on the relaxation model is stable. We propose two numerical discretizations that are respectively first and second order in time, but both second-order in space. The linear and nonlinear stability of the schemes is studied and it is demonstrated that the schemes are both discrete divergence free. We verify the numerical schemes using manufactured solutions and Mayergoyz’s moving-front solution in both one and two space dimensions, as well as on a magnetization of a typical HTS.

Keywords: high-temperature superconductors, Maxwell’s equations, FDTD method, Yee’s scheme, relaxation scheme, degenerate parabolic equation, $p$-curl problem, nonlinear diffusion, staggered grids, structured grids

2010 MSC: 35K92, 35Q61, 65M06

*We acknowledge financial support from National Science and Engineering Research Council of Canada.
1. Introduction

High-temperature superconductors (HTS) are found in various industrial applications such as cooling systems [1] and high current compact power cables [2]. With the commercial availability of low-cost HTS, engineers anticipate that many new high-power devices may be built from HTS, such as passive fault-current limiters [3], transformers, and stronger magnetic resonance imaging machines [4]. To design these new devices will require a better understanding of (i) the propagation of the magnetic front inside HTS and (ii) the appearance of hot-spots due to defects in fabrication, a process referred to as quenching [5]. These phenomena are critical in HTS, more so than for low-temperature superconductors (LTS), although quenching can still have catastrophic consequences [6]. This paper will focus on (i), since (ii) would involve somewhat separate issues related to heat transfer.

The simplest power-law model for electrical resistivity in a HTS leads to an eddy-current problem that is an analogue of the $p$-Laplacian [7]. The electrical engineering community has been studying this problem for over two decades [8], using mostly commercial softwares such as COMSOL Multiphysics® [9] and FlexPDE adapted partially to this problem. Unfortunately, the presence of sharp magnetic fronts and restrictive time-stepping imply that numerical solutions in 3-D remain too computationally expensive to permit the optimal design of new HTS devices. A few research groups, such as those of Dr. Brambilla in Milano, of Dr. Dutoit in Lausanne, and of Dr. Stenvall in Tampere, have developed their own research codes. Most numerical improvements have focused on reducing the number of degrees of freedom in 3-D, possibly through improved time step control with Sundials [10], adaptive mesh refinement and a posteriori error estimators [11, 12], symmetry considerations, but mostly with the help of multi-scale models [13]. In the vast majority of these developments from the engineering community, the discretization was obtained from a standard finite element (FE) formulation with explicit time-stepping, Nédélec elements or mixed elements with a divergence cleaning procedure. The literature contains formulations in the electric field [14], the magnetic field, and often in the potential variables, but we will focus on the $H$ formulation for the reasons described by Grilli and Brambilla [15]. Finally, we emphasize that within the engineering community, it is acknowledged that the power-law model is only an approximation of the correct electrical resistivity and that better models should be developed [16, 17, 18], hence future discretizations should be able to adapt to more general multi-scale electrical resistivity laws.

The main contribution of this paper is the construction and analysis of a relaxation approximation for the $p$-curl problem. The proposed relaxation approximation, or simply relaxation model, can be used as the basis for the construction of new discretizations, as the second author did in [19], the first author did for discontinuous Galerkin discretizations [20], and as we do for FDTD schemes in Section [4] and Section [5] of this work. The analysis of the staggered FDTD
scheme we propose, based on the relaxation model, will be shown to be stable and second-order in time and space, while the naive discretization of Yee applied to the $p$-curl produces an unstable scheme; see Section 4.4. The choice of a good relaxation model for a PDE is non-trivial affair. In fact, the relaxation model in [19] was inspired by Naldi and his colleagues approach [21] and an analogy to the Hamilton-Jacobi equation [22] but was more complex with two unphysical variables. In contrast, the relaxation model discussed here is similar to Jin’s model for nonlinear conservation laws [23], and therefore could be used with high-order finite-volume approximations, when applied in conjunction with the convection-diffusion splitting of Section 3.1. Furthermore, the structure of the relaxation model we propose suggests that many of the properties of the resulting scheme would be the same as those observed by Jin, namely the existence of a zero-relaxation limit and a natural IMEX time-stepping decomposition. The implicit treatment given to the diffusion term in our FDTD scheme could allow the straightforward treatment of diffusion transverse to the front [24, 17], which is currently neglected within the engineering community.

Relaxation models belong to a growing literature [25, 26, 27], motivated by the characterization of certain PDEs as macroscopic limits of kinetic models with non-equilibrium thermodynamics. Relaxation models have been developed for hyperbolic systems, and more recently for parabolic systems [28]. This work is inspired by the research of Naldi et al. on the porous media equation [29], yet the nonlinearity in the $p$-curl problem is significantly different than the one in the porous media equation since it involves powers of derivatives of primitives rather than of the primitives themselves. While the mathematical analysis of the macroscopic limit of our relaxation model is beyond the scope of this paper, we use an inner/outer layer analysis [30] and an expansion in $H$ to provide evidence that the relaxation model is a stable approximation of the $p$-curl and to relate this stability to anisotropic diffusion originating from the relaxation model. In our opinion, this is sufficient evidence for its use in numerical approximations. Hence, the study of the relaxation model in this paper allows its use as a foundational principle for the construction of future schemes.

As far as the existence of solutions to the $p$-curl problem is concerned, we refer the interested reader to the works of Yin [7, 31, 32] concerning well-posedness. Formal relations between solutions for the Bean model and the power-law model have also been examined [33]. Existence of solutions to finite-element method of lines approximations have been examined in [24, 86] but just outside of the ideal $W^p(\text{curl})$ functional framework. Janíková and Slodička studied, among other issues, implicit method of lines formulations with respect to the electric field [35]. Miranda and his colleagues have studied well-posedness for mixed boundary conditions [36]. Recent work by Prigozhin and Barrett demonstrates that a mixed finite-element approximation converges to solutions of the $p$-curl problem [37]. There have been a number of preliminary studies focusing on more advanced numerical topics [11], but there is still much to be done on the numerical solution to the $p$-curl problem, particularly by adapting the extensive literature on the parabolic $p$-Laplacian problem [38, 39].
The scheme is limited to structured grids and so, this might be considered a significant drawback to the method. Yet, HTS wires with rectangular cross-sections are formed of ceramics surrounded with layers of ordinary conductors, hence structured grids are not impractical for modelling several configurations of HTS wires [40]. This paper discusses only discretization in one and two space dimensions over rectangular grids, but there are obvious extensions to 3-D and to general structured grids, such as cylindrical or spherical grids. Experimental studies are done to characterize the exponent $p$ for different superconducting materials of type II and this exponent is typically found to be roughly 50 with a wide variance between 25 and 100 [11]. We note that some numerical methods have difficulties with values of $p$ above 10, such as the stability requirement of prohibitively small time steps. The relatively compact and explicit scheme described here has such small time steps but its efficiency compensates for this cost.

The relaxation model introduces a relaxation time $\epsilon$ which corresponds to sub-scale physics that are neglected in the $p$-curl problem. This is consistent with the power-law model of electrical resistivity which is itself a macroscopic limit of a hierarchy of models for superconductivity, such as the London model and vortex density models, which include physics for current carrying vortices [10]. The power-law model is already known to neglect transverse dissipation and alternative models, such as the elliptic critical-state model, are being developed [17]. The implementation of these more complex models into finite element models could be quite complex, but the relaxation model separates the sub-scale model from the convective process and hence can easily accommodate these with little to no effort. It is therefore natural to examine alternative formulations of the $p$-curl problem that model the multi-scale character of this problem.

In this paper, we introduce in Section 2 the $H$ formulation for the $p$-curl problem and the physical assumptions underlying it. In Section 3, the relaxation approximation for the $p$-curl problem is introduced and its stability is studied by means of inner/outer layer expansions and of a time-scale expansion with respect to only the magnetic field. The 1-D case is later studied in detail, including its truncation error and its stability. This article proposes two schemes, called respectively schemes A and B. Scheme A is stable and only first-order in time, yet is seen to be different from the unstable extension of Yee’s scheme to a nonlinear flux. In Section 4, the schemes A and B are described in 2-D and we show that its numerical solutions remain divergence-free if the initial data is also divergence-free, at least with respect to a common discretization of divergence [42]. Either stability in the $L^\infty$ norm or using the Von Neumann stability analysis is analyzed for both schemes and diffusive stability conditions are identified. Numerical tests in 1-D using Mayergoyz’s solution support our earlier analysis and are clearly seen to be free of spurious oscillations. Finally, Section 5 concludes the paper with three numerical results in 2-D. The first verifies both 2-D numerical schemes using a problem with a manufactured solution. The second describes a magnetic moving front generated by an external magnetic field increasing linearly in time along the boundary, similar to what is modelled in applications. The third numerical experiments are based on mag-
netization of a typical HTS wire and, besides demonstrating the effectiveness of the method on representative applications, also exhibit propagating magnetic fronts free of spurious oscillations.

2. The $p$-curl Problem

In this section we present the basic assumptions upon which the $p$-curl problem rests. We neglect the more technical details of the well-posedness of the $p$-curl problem first articulated in the work of Yin [7, 31].

The electric and magnetic fields in a domain $\Omega$ are modelled by Maxwell’s equations

$$
\begin{align*}
\frac{\partial}{\partial t} B + \nabla \times E &= 0, \\
\frac{\partial}{\partial t} D - \nabla \times H &= -J, \\
\nabla \cdot B &= 0, \\
\nabla \cdot D &= \rho_c,
\end{align*}
$$

where $B$ is the magnetic induction field, $E$ is the electric field, $D = \epsilon \epsilon_0 E$ is the electric field displacement, $H = \mu^{-1} B$ is the magnetic field, $\rho_c$ is the electric charge density, $\epsilon_\epsilon_0$ is the electric permittivity, $\mu$ is the magnetic permeability and $J$ is the current density. The equations of system (1) are respectively Faraday’s law, Ampère-Maxwell’s law and Gauss’ law. By Ohm’s law, we have

$$
E = \rho \left( J + J_c \right)
$$

where $\rho$ is called the electrical resistivity and $J_c$ is the applied current density. We assume that $J_c = 0$. We also assume $|\partial_t D| \ll |J|$ which implies the magneto-quasistatic condition [39], i.e. $\partial_t D \approx 0$. Under those assumptions, we obtain a nonlinear eddy-current problem in $H$ formulation

$$
\begin{align*}
\frac{\partial}{\partial t} (\mu H) + \nabla \times \left( \rho \left| \nabla \times H \right| \nabla \times H \right) &= 0, \\
\nabla \cdot (\mu H) &= 0.
\end{align*}
$$

In a continuous HTS model, the electrical resistivity is given by a power-law

$$
\rho \left| \nabla \times H \right| = \frac{E_c}{J_c^{p-1}} \left| \nabla \times H \right|^{p-2}
$$

where $|\cdot|$ is the Euclidean norm, $J_c$ is the critical current density of the HTS, $E_c$ is the electric field amplitude at the critical current, and $p$ is an experimentally determined parameter that is usually between 50 and 100 [44]. Early models implemented Bean’s model [45]

$$
|E| = \left\{ \begin{array}{ll}
0 & \text{if } |J| < J_c, \\
e & \text{if } |J| = J_c,
\end{array} \right.
$$

with $e \in [0, \infty]$, which is a valid approximation when $p$ is very large [32]. In our experience, most numerical schemes begin to struggle when $p \geq 10$ and their computational efficiency and accuracy may deteriorate for large values of $p$ [4].
Combining the first equation of (3) with the power-law model (4) leads to
\[ \partial_t (\mu H) + \nabla \times (\alpha |\nabla \times H|^{p-2} \nabla \times H) = 0 \] (6)
where \( \alpha = E_c/J_p^{-1} \). Equation (6) is the nonlinear parabolic equation known as the \( p \)-curl problem and named in this way to emphasize its similarity to the parabolic \( p \)-Laplacian [46, 31].

Assuming that the boundary \( \partial \Omega \) is subdivided into disjoint subsets \( \Gamma_N \) and \( \Gamma_D \neq \emptyset \), and that two functions \( f, g \) are known on \( \Gamma_D \) and \( \Gamma_N \) respectively, then we search for a solution \( H \) of (6) that satisfies
\[ n \times H = f \quad \text{on } \Gamma_D \times [0, T], \quad (7) \]
\[ n \times (\rho(J)J) = g \quad \text{on } \Gamma_N \times [0, T], \quad (8) \]
where \([0, T]\) is a time interval with \( T > 0 \) and \( n \) is the outward unit normal to \( \partial \Omega \). By Ohm’s law, the second boundary condition can in fact be written \( n \times E = g \). In this paper, we will consider only boundary conditions (7) but more general boundary conditions, and their associated well-posedness theory, were considered by Miranda et al. [30].

3. Relaxation Model for the \( p \)-curl Problem

The relaxation model for the \( p \)-curl is one of the two main contributions of this paper because it can be used to construct new numerical discretization to the \( p \)-curl problem. Eventually in Section 4 we will construct a stable scheme similar to Yee’s scheme but that could not have been identified without the relaxation model. Other numerical discretizations based on the relaxation model are being studied [19, 20]. As a foundational principle, it is essential that the relation between the \( p \)-curl problem (6) and the relaxation approximation below be thoroughly studied.

The typical ansatz for the construction of a relaxation approximation [23] is to introduce a new time scale \( \epsilon \) over which the solutions of an initial model, linear in our case, relax over a time scale \( \epsilon \) towards the original nonlinear equilibrium model. Typically, the time scale \( \epsilon \) is made shorter than the time step \( \Delta t \) in any discretization and hence introduces phenomena that will remain under-resolved by any numerical discretization. It is therefore essential to study the stability of the relaxation model and to analyze the macroscopic limit as \( \epsilon \to 0 \) of the relaxation model to the original model. In this section, these analysis will be pursued with the help of an inner/outer layer expansion and an expansion similar to the Chapman-Enskog expansion, which will provide us with convincing evidence, albeit not a completely rigorous proof, of the stability and convergence of this approximation.

Before proceeding, a few remarks are in order on the physical relevance of a relaxation approximation for the \( p \)-curl. The power-law model (4) is known to neglect diffusion transverse to \( \nabla \times H \) and only appears as the simplest model, after Bean’s discontinuous model (5), of a hierarchy of models. Replacements
to the power-law model (4) are still actively sought [47, 17] and will require a better understanding of the sub-scale physics involved in the creation/loss of magnetic vortices inside the current carrying region of the HTS. It can therefore be said that the introduction of a relaxation approximation is natural for this problem and intended as a temporary adjunct for a physically motivated multiscale model. In any case, the relaxation approximation leads to discretizations that could be quickly adapted to other electrical resistivity models, something FE discretizations would struggle to do.

We propose the following straightforward relaxation model which separates the nonlinearity in (6) from the convective term

\[
\partial_t (\mu H) + \nabla \times E = 0, \\
\partial_t E - a^2 \nabla \times H = - \frac{1}{\epsilon} (E - \rho(|\nabla \times H|) \nabla \times H),
\]

(9)

where \(a\) is a positive constant and \(\epsilon\) is the relaxation parameter. We observe that if the initial data is divergence free, then the first equation in (9), implies that it remains divergence free at all time, thus allowing us to postpone the discussion of the constraint \(\nabla \cdot (\mu H) = 0\) until the discretization is introduced. The parameter \(a\) will be chosen to satisfy any stability condition we may eventually identify. Besides the obvious analogy in (2), a priori there is nothing suggesting that the auxiliary variable \(E\) should be interpreted as an electric field. Nevertheless, for convenience \(E\) will be called the electric field. We remark that the negative sign in front of the \(\epsilon^{-1}\) term is necessary because, after retaining only terms in \(E\) in the second equation of (9), one finds an exponentially decreasing field.

We begin our study of the relaxation model (9) with an inner/outer layer expansion, following [23] and [26], which will show that the transients at the \(\epsilon^{-1}\) time scale are exponentially decreasing. A second expansion with respect to \(\epsilon\), which we will call the \(\psi\)-expansion, will show that the solution of the relaxation model is a solution to a high-order perturbation of the \(p\)-curl problem. Such an expansion is typically done for kinetic models and allows one to determine its higher-order moments [48], and in those cases it is referred to as a Chapman-Enskog expansion. The \(\psi\)-expansion will provide further evidence that (9) is a stable model, by showing that the higher-order terms are coercive and locally monotone. For the sake of completeness, these terms will also be compared to \(p\)-curl operator and the linearization of the \(p\)-curl, which is usually referred to as the quasi-norm [49, 38].

We now begin the analysis with the inner/outer layer expansion. Assume that solutions to (9) are at least three times differentiable and that \(\mu\) is constant. We begin by studying the outer layer

\[
H = H_0 + \epsilon H_1 + O(\epsilon^2), \\
E = E_0 + \epsilon E_1 + O(\epsilon^2),
\]

(10)

which identifies the best fit of \(H\) and \(E\), independent of \(\epsilon\). Such an expansion cannot therefore approximate transients in time that scale with \(\epsilon\). This will
be done later with an inner layer analysis with terms depending, in a first approximation, on \( \tau = t/\epsilon \).

Substituting (10) into (9), writing \( P(J) = \rho(J)J \), we find
\[
\begin{align*}
\mu \partial_t (\mathbf{H}_0 + \epsilon \mathbf{H}_1) + \nabla \times (\mathbf{E}_0 + \epsilon \mathbf{E}_1) &= \mathcal{O}(\epsilon^2), \\
\partial_t (\mathbf{E}_0 + \epsilon \mathbf{E}_1) - a^2 (\nabla \times \mathbf{H}_0 + \epsilon \nabla \times \mathbf{H}_1) &= -\frac{1}{\epsilon} \left( \mathbf{E}_0 + \epsilon \mathbf{E}_1 - P(\nabla \times \mathbf{H}_0) - \epsilon dP(\nabla \times \mathbf{H}_0) \nabla \times \mathbf{H}_1 \right) + \mathcal{O}(\epsilon),
\end{align*}
\]

(11)

where \( dP \) is the Jacobian from the Taylor series expansion
\[
P(\nabla \times \mathbf{H}_0 + \epsilon \nabla \times \mathbf{H}_1 + \mathcal{O}(\epsilon^2)) = P(\nabla \times \mathbf{H}_0) + \epsilon dP(\nabla \times \mathbf{H}_0) \nabla \times \mathbf{H}_1 + \mathcal{O}(\epsilon^2).
\]

Grouping terms of same order, we find the hierarchy
\[
\begin{align*}
\mathcal{O}(\epsilon^{-1}) : & \quad \mathbf{E}_0 = P(\nabla \times \mathbf{H}_0), \\
\mathcal{O}(1) : & \quad \begin{cases}
\mu \partial_t \mathbf{H}_0 + \nabla \times \mathbf{E}_0 = 0, \\
\partial_t \mathbf{E}_0 - a^2 \nabla \times \mathbf{H}_0 = -(\mathbf{E}_1 - dP(\nabla \times \mathbf{H}_0) \nabla \times \mathbf{H}_1),
\end{cases} \\
\mathcal{O}(\epsilon) : & \quad \mu \partial_t \mathbf{H}_1 + \nabla \times \mathbf{E}_1 = 0,
\end{align*}
\]

(13)

(14)

(15)

where we have ignored the equation for \( \partial_t \mathbf{E}_1 \) since it would involve the Hessian of \( P, \mathbf{E}_2 \) and \( \nabla \times \mathbf{H}_2 \). The identities (13) to (15) will be used immediately to simplify the inner layer expansion.

We assume that there exists a time scale \( t = \tau \epsilon \), which vanishes as \( \epsilon \to 0 \), over which transients occur and are therefore not captured by \( \mathbf{E}_1 \) and \( \mathbf{H}_1 \). A multiple time scale analysis would assume that the characteristic time scale for these transients could be written \( t = \tau_1 \epsilon + \tau_2 \epsilon^2 + \ldots \), but the first time scale will be sufficient. Inversely, for fixed \( t \) and vanishing \( \epsilon \), the time scale \( \tau \to \infty \) hence the behaviour of the transients for small \( \epsilon \) correspond to limits as \( \tau \to \infty \) of the inner layer. Assume that the solution to (9) has solutions which can be written in the form
\[
\begin{align*}
\mathbf{H}(x, t) &= (\mathbf{H}_0(x, t) + \epsilon \mathbf{H}_1(x, t) + \mathcal{O}(\epsilon^2)) + (\dot{\mathbf{H}}_0(x, \tau) + \epsilon \dot{\mathbf{H}}_1(x, \tau) + \mathcal{O}(\epsilon^2)), \\
\mathbf{E}(x, t) &= (\mathbf{E}_0(x, t) + \epsilon \mathbf{E}_1(x, t) + \mathcal{O}(\epsilon^2)) + (\dot{\mathbf{E}}_0(x, \tau) + \epsilon \dot{\mathbf{E}}_1(x, \tau) + \mathcal{O}(\epsilon^2)),
\end{align*}
\]

where the terms in the first and second parentheses represent respectively the outer and the inner layer. Substituting these into (9) and observing that \( \partial_t f(x, \tau) = \frac{1}{\epsilon} \partial_x f(x, \tau) \), we find
\[
\begin{align*}
\mu \partial_t (\mathbf{H}_0 + \epsilon \mathbf{H}_1) + \frac{\mu}{\epsilon} \partial_x (\mathbf{H}_0 + \epsilon \dot{\mathbf{H}}_1) + \nabla \times (\mathbf{E}_0 + \epsilon \mathbf{E}_1 + \dot{\mathbf{E}}_0 + \epsilon \dot{\mathbf{E}}_1) &= \mathcal{O}(\epsilon^2), \\
\partial_t (\mathbf{E}_0 + \epsilon \mathbf{E}_1) + \frac{1}{\epsilon} \partial_x (\dot{\mathbf{E}}_0 + \epsilon \dot{\mathbf{E}}_1) - a^2 \nabla \times (\mathbf{H}_0 + \epsilon \mathbf{H}_1 + \dot{\mathbf{H}}_0 + \epsilon \dot{\mathbf{H}}_1) &= -\frac{1}{\epsilon} \left( (\mathbf{E}_0 + \epsilon \mathbf{E}_1 + \dot{\mathbf{E}}_0 + \epsilon \dot{\mathbf{E}}_1 + \mathcal{O}(\epsilon^2)) - P(\nabla \times (\mathbf{H}_0 + \epsilon \mathbf{H}_1 + \dot{\mathbf{H}}_0 + \epsilon \dot{\mathbf{H}}_1) + \mathcal{O}(\epsilon^2)) \right) + \mathcal{O}(\epsilon).
\end{align*}
\]
Isolating as before the terms of same order in \( \epsilon \), one finds

\[
\mathcal{O}(\epsilon^{-1}) : \begin{cases} 
\mu \partial_r \hat{H}_0 = 0, \\
\partial_r \hat{E}_0 = - \left( E_0 + \hat{E}_0 - P(\nabla \times (H_0 + \hat{H}_0)) \right), \\
\mu \partial_r \hat{H}_1 + \nabla \times (E_0 + \hat{E}_0) = 0,
\end{cases}
\tag{16}
\]

\[
\mathcal{O}(1) : \begin{cases} 
\partial_t E_0 + \partial_r \hat{E}_1 - a^2 \nabla \times (H_0 + \hat{H}_0) \\
\partial_t \hat{E}_0 = - \left( E_1 + \hat{E}_1 - dP(\nabla(H_0 + \hat{H}_0)) \nabla \times (H_1 + \hat{H}_1) \right),
\end{cases}
\tag{17}
\]

\[
\mathcal{O}(\epsilon) : \mu \partial_t \hat{H}_1 + \nabla \times (E_1 + \hat{E}_1) = 0. \tag{18}
\]

These equations for \( \hat{E}_0, \hat{E}_1, \hat{H}_0 \) and \( \hat{H}_1 \) can be simplified if we use the previous equations (13) to (15). We assume, as did Jin [23], that the initial transient \( \hat{H}_0(x, \tau) = 0 \) because the relaxation is effectively with respect to \( E \). Using the first equation of (16), we deduce immediately that

\[
\hat{H}_0(x, \tau) = 0, \quad \forall \tau \geq 0.
\]

This identity and (13) imply that the second equation of (16) becomes

\[
\partial_r \hat{E}_0 = - \hat{E}_0.
\]

This equation has solutions which decrease exponentially with respect to \( \tau \). Using now the first equation of (14) in the first equation of (17), we find the following equation

\[
\partial_r \hat{H}_1 + \nabla \times \hat{E}_0 = 0.
\]

Given that \( \hat{E}_0 \) is already exponentially decreasing, so will \( \nabla \times \hat{E}_0 \) and \( \hat{H}_1 \). We obtain an equation for the last variable \( \hat{E}_1 \) by substituting (15) into (18) and find

\[
\partial_r \hat{E}_1 = -(\hat{E}_1 - dP(\nabla \times H_0) \nabla \times \hat{H}_1).
\tag{19}
\]

The fact that \( \hat{H}_1 \) is exponentially decreasing implies that \( dP(\nabla \times H_0) \nabla \times \hat{H}_1 \) acts as a source term for the evolution equation \( \partial_r \hat{E}_1 = - \hat{E}_1 \). We summarize the analysis in the following theorem.

**Theorem 3.1.** On a domain \( \Omega \subset \mathbb{R}^3 \), consider \( C^3(\Omega) \) solutions \( E \) and \( H \) of (9) that are also at least \( C^2 \) with respect to the relaxation scale \( \epsilon \). It follows that the terms in the expansion (inner/outer) are \( \hat{H}_0 = 0 \) with \( \hat{E}_0, \hat{E}_1, \hat{H}_1 \) exponentially decreasing over a time scale \( \epsilon \). Moreover, the expansion (10) satisfies the equation

\[
\mu \partial_t \hat{H} + \nabla \times P(\nabla \times H) = -\epsilon a^2 \nabla \times \nabla \hat{H}
\]

\[
- \frac{\epsilon}{\mu} \nabla \times \left[ \rho(\nabla \times H)(I + (p-2)\mathbf{n}_\nabla \otimes \mathbf{n}_\nabla) \nabla \times P(\nabla \times H) \right] + \mathcal{O}(\epsilon^2),
\tag{20}
\]

where \( \mathbf{n}_\nabla = \nabla \times H / |\nabla \times H| \).
Proof. Consider the first equation in (9) and substitute in the expansion (10) for $E$ to obtain

$$\partial_t (\mu H) + \nabla \times (E_0 + \epsilon E_1) = O(\epsilon^2).$$

Doing the same with the second equation in (9), we find

$$\partial_t (E_0 + \epsilon E_1) - a^2 \nabla \times H = -\frac{1}{\epsilon} (E_0 + \epsilon E_1 - P(\nabla \times H)) + O(\epsilon),$$

and we deduce two identities from the $\epsilon^{-1}$ and $O(1)$ terms

$$E_0 = P(\nabla \times H),$$

$$E_1 = -\partial_t E_0 + a^2 \nabla \times H.$$

Substituting the second identity into (21), we obtain

$$\partial_t (\mu H) + \nabla \times P(\nabla \times H) = \epsilon (-a^2 \nabla \times \nabla \times H + \nabla \times \partial_t P(\nabla \times H)) + O(\epsilon^2).$$

This last equation is interesting in its own right and will be useful later, hence it will be collected in a remark below. Returning to the expansion, we remove the time derivative on $\nabla \times \partial_t P(\nabla \times H)$ by computing

$$\partial_t P(\nabla \times H) = \rho(|\nabla \times H|) \partial_t (\nabla \times H) + \partial_t \left( \rho(|\nabla \times H|) \right) \nabla \times H.$$

The second term in Leibniz’s formula needs to be computed explicitly

$$\partial_t \rho(|\nabla \times H|) = \alpha (p-2) |\nabla \times H|^{p-3} \frac{1}{|\nabla \times H|^T} (\nabla \times H)^T \partial_t \nabla \times H,$$

and, using the short-hand $n_{\nabla} := \nabla \times H / |\nabla \times H|$ for the unit vector in the direction of $\nabla \times H$, we conclude

$$\partial_t \left( \rho(|\nabla \times H|) \right) \nabla \times H = \alpha (p-2) |\nabla \times H|^{p-2} (\nabla_t \nabla \times H)^T n_{\nabla} \otimes n_{\nabla} \partial_t \nabla \times H,$$

where we have written $n_{\nabla} \otimes n_{\nabla}$ to denote the square rank one matrix $n_{\nabla} (n_{\nabla})^T$. We consider a last simple identity obtained by substituting $E = P(\nabla \times H) + O(\epsilon)$ into the first equation of (9)

$$\nabla \times \partial_t H = -\mu^{-1} \nabla \times (\nabla \times E) = -\mu^{-1} \nabla \times \left( \nabla \times (\rho(|\nabla \times H|) \nabla \times H) \right) + O(\epsilon).$$

We are now in a position to complete the perturbation expansion by substituting (24)-(26) into (23),

$$\partial_t (\mu H) + \nabla \times P(\nabla \times H)$$

$$= \epsilon (\nabla \times \partial_t P(\nabla \times H) - a^2 \nabla \times \nabla \times H) + O(\epsilon^2)$$

$$= O(\epsilon^2) - \epsilon a^2 \nabla \times \nabla \times H$$

$$- \epsilon \mu^{-1} \nabla \times \left[ \rho(|\nabla \times H|) \right]$$

$$\left( I + (p-2) n_{\nabla} \otimes n_{\nabla} \right) \nabla \times \left( \nabla \times (\rho(|\nabla \times H|) \nabla \times H) \right).$$
Remark 3.2. The expansion \((10)\) identifies the outer layer while the inner layer is known to contribute exponentially decreasing transients. Stability of the relaxation model is therefore determined by the complex nonlinear term on the right hand side of the \(H\)-expansion. Unfortunately, at the moment we are only able to show that this term is coercive, with a proof of monotonicity still out of reach.

Without spelling out the specific Banach space \(F_p\) involved, the fourth-order nonlinear term in \((20)\) defines an operator

\[
\mathcal{E}_p : F_p \rightarrow F_p^*,
\]

\[
u \mapsto \epsilon \mu^{-1} (\nabla \times (|\nabla \times \upsilon|) \nabla \times \upsilon, 
\nabla \times \left( I + (p - 2) n_\nabla \otimes n_\nabla \right) \rho(|\nabla \times \upsilon|) \nabla \times \varphi),
\]

where \(\varphi\) is a test function in \(F_p\), and \(\langle \cdot, \cdot \rangle\) is the pairing by integration. Taking \(\varphi = \upsilon\) and performing integration by parts (assuming homogeneous boundary conditions) we find

\[
\epsilon \mu^{-1} (p - 1) (\nabla \times (|\nabla \times \upsilon|) \nabla \times \upsilon, \nabla \times (|\nabla \times \upsilon|) \nabla \times \upsilon) \geq 0,
\]

(27)

where we have used the simple identity

\[
(I + (p - 2) n_\nabla \otimes n_\nabla) \rho(|\nabla \times \upsilon|) \nabla \times \upsilon = (p - 1) \rho(|\nabla \times \upsilon|) \nabla \times \upsilon.
\]

In fact, it is possible to show that this term vanishes if and only if \(\upsilon\) vanishes. Hence \(\mathcal{E}_p\) is a strictly coercive operator.

Remark 3.3. We conclude this section by relating a term in the inner/outer layer expansion to a well-known linearization term appearing in asymptotically exact error estimators of the \(p\)-Laplacian. This term was identified by Alaoui, Ern and Vohralík as a linearization of Newton-Raphson type for the flux of the \(p\)-iteration; see equation (2.15) in [49]. In our case, it appears as a term on the right-hand side of \((17)\) and \((19)\), namely

\[
dP (\nabla \times H_0) \nabla \times H_1 = \rho(|\nabla \times H_0|) \nabla \times H_1 + \frac{\rho'(|\nabla \times H_0|)}{|\nabla \times H_0|} n_\nabla \otimes n_\nabla \nabla \times H_1,
\]

where we have used a calculation similar to the one used for \((25)\), and in particular \(n_\nabla := \nabla \times H_0/|\nabla \times H_0|\). The projection \(n_\nabla \otimes n_\nabla\) in the direction of \(\nabla \times H_0\) implies that this term in \((19)\) applies anisotropic diffusion on the current, thus stabilizing it.

In conclusion, this section has provided an analysis of the stability of the relaxation approximation \((9)\) to the \(p\)-curl problem \((6)\). Our analysis has identified the transients in the approximation that will become under-resolved by a time discretization of the problem. The analysis, while under strict regularity conditions, is done with the same level of rigour as [23] and does indicate that the relaxation model could be used numerically to approach solutions to the \(p\)-curl.
We remark that this expansion shows that no stability condition, such as Liu’s subcharacteristic condition, is required to ensure that the relaxation model approximates the $p$-curl problem, as long as the discretization satisfies $\Delta t, \Delta x \gg \epsilon$, which we call the under-resolved regime.

3.1. Splitting into Convective and Diffusive Terms

Relaxation models often naturally lead to splittings into convective and diffusive terms [23, 29] which can be used to produce numerical discretizations of IMEX type. This splitting will not be used in our later construction of a FDTD scheme. Nevertheless, the relaxation model could be used as the basis for many other schemes, in particular high-order explicit schemes for convection could be combined with implicit formulations for the diffusion [19, 20].

Consider the relaxation model (9) decoupled into a linear convective step and a nonlinear diffusive step. The convective step is

$$\frac{\partial}{\partial t} (\mu H) + \nabla \times E = 0,$$
$$\frac{\partial}{\partial t} E - a^2 \nabla \times H = 0,$$

and the nonlinear diffusive step is

$$\frac{\partial}{\partial t} (\mu H) = 0,$$
$$\frac{\partial}{\partial t} E = -\frac{1}{\epsilon} \left( E - \rho (|\nabla \times H|) \nabla \times H \right).$$

(28)

(29)

It is easy to show that the convective step, equations (28), forms a linear hyperbolic system with eigenvalues $\pm a \mu^{-1/2}$ and 0. The hyperbolicity implies that many numerical methods, in particular high-order methods, could be used to propagate fronts without introducing spurious oscillations. Finally, we remark that when implementing this splitting, with explicit treatment of (28) and implicit treatment of (29), the actual solve of (29) could be done explicitly [28] thus leading to a simple robust scheme.

4. 1-D Staggered Discretization

This section contains a detailed description of two discretizations of the relaxation model over a staggered mesh. We derive in Section 4.1 a first discretization, second-order in space, but first-order in time, which we call Scheme A. Later, in Section 4.5, we also introduce a correction that allows us to obtain a fully second-order scheme, which we call Scheme B. We compute the truncation error and perform a stability analysis for both proposed schemes. For both proposed schemes, we introduce a version with $\epsilon > 0$, which we call the relaxation scheme, and a version with $\epsilon = 0$, which we call the relaxation limit of the scheme. The discretization in two space dimensions is briefly described in Section 5.

By considering an infinite HTS slab, the electric field and the magnetic field can be written respectively as $E(x, t) = (0, 0, E(x, t))$ and $H(x, t) =$
(0, H(x,t), 0). The divergence free condition on the magnetic field is always satisfied in 1-D for a constant magnetic permeability. Those assumptions lead to the equation

$$\partial_t (\mu H) - \partial_x (\mu \frac{\partial_x H}{|\partial_x H|^{p-2}} \partial_x H) = F$$

which is the p-Laplacian problem [50] with a source term F. The relaxation approximation is then given by

$$\partial_t (\mu H) - \partial_x E = F,$$

$$\partial_t E - a^2 \partial_x H = -\frac{1}{\epsilon} (E - \rho(\partial_x H) \partial_x H).$$

We briefly present the H-expansion in 1-D since in this case, it is much simpler to understand. The same procedure as in Section 3 demonstrates that

$$\partial_t (\mu H) - \partial_x \left( \rho(\partial_x H) \partial_x H \right) - \epsilon a^2 \partial_x^2 H$$

$$= -\epsilon\mu^{-1} (p - 1) \partial_x \left( \rho(\partial_x H) \partial_x H \right) + O(\epsilon^2).$$

The same proof for inequality [27] applies here and shows that the fourth-order term is coercive. Even in the simpler form given above, we have been unable to demonstrate monotonicity.

### 4.1. Numerical Scheme

In this section, we describe a staggered discretization of the scheme in 1D that is only first-order in time, which will be referred to as Scheme A. An improvement that is second-order in time will be presented in Section 4.5.

Consider a domain \( \Omega = [a_L, a_R] \) subdivided into \( N_x \) equidistant cells such that \( \Omega_i = [x_{i-1/2}, x_{i+1/2}] \) for \( i = 1, 2, \ldots, N_x \) forms the \( i \)-th cell of length \( \Delta x := (a_R - a_L)/N_x \). We define the node \( x_{i+1/2} := a_L + i \Delta x \) for \( i = 0, 1, \ldots, N_x \) and the cell center \( x_i := a_L + (i - 1/2) \Delta x \) for \( i = 1, \ldots, N_x \). For time discretization, we consider a time interval \( I = [0, T] \) subdivided into \( N_t \) equidistant subintervals of length \( \Delta t := T/N_t \). Since we stagger the discretization both in space and time, we will need to introduce \( t_n := n \Delta t \) for \( n = 0, 1, \ldots, N_t \) and \( t_{n+1/2} = (n + 1/2) \Delta t \) for \( n = 0, \ldots, N_t - 1 \).

In order to discretize the relaxation model [31], we mimic Yee’s scheme [51] by staggering the variables both in space and time. The magnetic field \( H \) is defined at the nodes and the time \( t_{n+1/2} \) while the electric field \( E \) is defined at the cell center and the time \( t_n \). This ansatz is motivated by the integral properties of electromagnetics. The relaxation model [31] discretized with second-order centered finite differences leads to the following centered scheme

$$\mu \frac{H_{i+1/2}^{n+1/2} - H_{i+1/2}^{n-1/2}}{\Delta t} - \frac{E_{i+1}^n - E_{i}^n}{\Delta x} = F_{i+1/2}^n,$$

$$\frac{E_{i+1}^{n+1} - E_{i+1}^{n}}{\Delta t} - a^2 \frac{H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2}}{\Delta x} = -\frac{1}{\epsilon} \left( \frac{E_{i+1}^{n+1} + E_{i}^{n}}{2} - \rho_{i+1/2}^{n+1/2} \frac{H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2}}{\Delta x} \right).$$

(32)
where $F^n_{i+1/2} = F(x_{i+1/2}, t_n)$ and

$$\rho^n_{i+1/2} = \alpha \left| \frac{H^n_{i+1/2} - H^n_{i-1/2}}{\Delta x} \right|^{p-2}.$$ 

In Section 4.4, its relaxation limit when $\epsilon \to 0$ will be related to a direct application of Yee’s scheme and it can be shown that scheme (32) possesses an unstable relaxation limit, which is no surprise since centered schemes are known to be unstable for stiff problems. In order to obtain a scheme with a stable relaxation limit, we propose to modify the scheme (32) as

$$\mu \frac{H^n_{i+1/2} - H^n_{i+1/2}}{\Delta t} - \frac{E^n_{i+1} - E^n_{i}}{\Delta x} = F^n_{i+1/2},$$

(33)

$$\frac{E^n_{i+1} - E^n_{i}}{\Delta t} - \frac{H^n_{i+1/2} - H^n_{i-1/2}}{\Delta x} = -\frac{1}{\epsilon} \left( E^n_{i+1} - \rho^n_{i+1/2} \frac{H^n_{i+1/2} - H^n_{i-1/2}}{\Delta x} \right).$$

Although having a stable relaxation limit (see Section 4.3), scheme (33) is only first-order in time as it will be shown in the next subsection. The schemes given by (33) and (32) are also known as relaxation schemes when $\epsilon > 0$. Both relaxation schemes above (for $\epsilon > 0$) were virtually indistinguishable in numerical experiments.

As Naldi et al. [29], we consider the relaxation limit of the Scheme A (33), i.e. the limiting case when $\epsilon$ tends to zero. The relaxation limit of Scheme A is clearly

$$H^n_{i+1/2} = H^n_{i+1/2} + \frac{\Delta t}{\mu} \left( E^n_{i+1} - E^n_{i} \right) + \frac{\Delta t}{\mu} F^n_{i+1/2},$$

(34)

$$E^n_{i+1} = \rho^n_{i+1/2} \frac{H^n_{i+1/2} - H^n_{i-1/2}}{\Delta x}.$$  

Note the obvious $\Delta t$ error in the second equation, since the right-hand side and the left-hand side are centered at different times.

For the initial conditions, we impose $H$ at $t_{1/2} = \Delta t/2$ and $E$ at $t_0 = \Delta t$. If the electric field $E$ is not available, we can compute it from the second equation of (34).

4.2. Truncation Error Analysis

Our purpose is to study the truncation error of Scheme A and in particular to evaluate it at the relaxation limit. We demonstrate that when $\epsilon > 0$, Scheme A is first-order in time and second-order in space. The relaxation limit of Scheme A is found to be a consistent approximations of the $p$-curl problem, and we will see later that it possesses a stable relaxation limit. For simplicity, we will assume throughout that $F = 0.$
Using the appropriate Taylor expansions about \((x_{i+1/2}, t_n)\) for the first equation above, we find

\[
\mu \partial_t H(x_{i+1/2}, t_n) - \partial_x E(x_{i+1/2}, t_n) = -\mu \frac{\Delta t^2}{24} \partial_t^3 H(x_{i+1/2}, t_n) + \frac{\Delta x^2}{24} \partial_x^2 E(x_{i+1/2}, t_n) + O(\Delta t^3 + \Delta x^3). \tag{35}
\]

Recalling \(P(\partial_x H) = \rho(\partial_x H) \partial_x H\), the truncation error with respect to \(\Delta t\) and \(\Delta x\) for the second equation of the system \((33)\) given by

\[
E(x_i, t_{n+1/2}) - P(\partial_x H(x_i, t_{n+1/2})) = -\epsilon \left( \partial_t E(x_i, t_{n+1/2}) - a^2 \partial_x H(x_i, t_{n+1/2}) \right) - \epsilon \frac{\Delta t^2}{4} \partial_x^2 E(x_i, t_{n+1/2}) + \epsilon a^2 \frac{\Delta x^2}{4} \partial_x^2 H^{n+1/2}
\]
\[+ O(\Delta t) + O(\Delta x^2 + \epsilon \Delta t^3 + \epsilon \Delta x^2) \]

where we have multiplied the expansion by \(\epsilon\), to ensure that the variable \(E\), with respect to which convergence is measured, appears as a zero-th order term, and \(E^{n+1} = E_i^{n+1/2} + O(\Delta t)\). The truncation error for Scheme A is \(O(\Delta t + \Delta x^2 + \epsilon (\Delta t^2 + \Delta x^2))\).

We now study the truncation error at the relaxation limit. This leads to the discretized form of \((30)\),

\[
\frac{\mu}{\Delta t} \left( H_i^{n+1/2} - H_i^{n-1/2} \right) = \frac{1}{\Delta x^2} \left( \rho_i^{n-1/2} H_i^{n-1/2} - (\rho_{i+1}^{n-1/2} + \rho_i^{n-1/2}) H_{i+1/2}^{n-1/2} + \rho_i^{n-1/2} H_{i-1/2}^{n-1/2} \right). \tag{36}
\]

Notice that the left-hand side is centered at time \(t_n\) while the right-hand side is centered at time \(t_{n-1/2}\). Hence, the Taylor series expansions introduce a first order error in time

\[
\mu \partial_t H(x_{i+1/2}, t_n) - \partial_x P(\partial_x H(x_{i+1/2}, t_n)) = O(\Delta t) + O(\Delta t^2 + \Delta x^2),
\]

which still implies that Scheme A has a relaxation limit that is consistent with the \(p\)-curl problem. We summarize the results of this section in the following theorem.

**Theorem 4.1.** For the relaxation Scheme A given by \((33)\), the truncation error is

\[
O(\Delta t + \Delta x^2 + \epsilon (\Delta t^2 + \Delta x^2))
\]

Taking the limit when \(\epsilon \to 0\), Scheme A possesses a relaxation limit, scheme \((34)\), with a truncation error

\[
O(\Delta t) + O(\Delta x^2).
\]

The truncation error shows that there exists two regimes of convergence. Following Jin \cite{23}, the first is called over-resolved and occurs when \(\epsilon\) is fixed but \(\Delta t, \Delta x < \epsilon\), while the second is called under-resolved and is observed when \(\Delta t, \Delta x \gg \epsilon\).
4.3. Stability Analysis

In this section, we demonstrate using a maximum norm stability analysis the existence of a criterion on the time step in order for the relaxation limit of Scheme A to be stable.

**Theorem 4.2.** Consider the 1-D relaxation approximation (31) of the p-curl with $F = 0$ and the power-law $P(J) = \rho(J) J$ with $p \geq 2$. If the time step satisfies

$$\Delta t \leq \frac{\mu \Delta x^2}{2P'_{\max}},$$

where $P'_{\max} = \max_{i} P'(J_i^{n-1/2})$, then the relaxation limit of Scheme A is stable in the sense that

$$\|H^{n+1/2}\|_\infty \leq \|H^{n-1/2}\|_\infty.$$

In other words, scheme (34) is stable in $l^\infty$.

**Proof.** We combine both equations of (34). This leads to

$$H_{i+1/2}^{n-1/2} = H_{i+1/2}^{n-1/2} + \frac{\Delta t}{\mu \Delta x^2} \left(P(J_{i+1}^{n-1/2}) - P(J_{i}^{n-1/2})\right)$$

where $J_{i}^{n-1/2} = \frac{H_{i+1/2}^{n-1/2} - H_{i-1/2}^{n-1/2}}{\Delta x}$. By the mean value theorem, there exists $\xi_i$ strictly between $J_{i}^{n-1/2}$ and $J_{i+1}^{n-1/2}$ such that

$$P(J_{i+1}^{n-1/2}) - P(J_{i}^{n-1/2}) = P'(\xi_i) (J_{i+1}^{n-1/2} - J_{i}^{n-1/2}).$$

For a given time $t_{n-1/2}$, the substitution of equation (38) into equation (37) gives

$$H_{i+1/2}^{n+1/2} = H_{i+1/2}^{n-1/2} + \frac{\Delta t P'(\xi_i)}{\mu \Delta x^2} \left(H_{i+1/2}^{n-1/2} - 2H_{i+1/2}^{n-1/2} + H_{i-1/2}^{n-1/2}\right),$$

which leads to

$$|H_{i+1/2}^{n+1/2}| \leq \frac{\Delta t P'(\xi_i)}{\mu \Delta x^2} |H_{i+1/2}^{n-1/2}| + \left|1 - \frac{2 \Delta t P'(\xi_i)}{\mu \Delta x^2}\right| |H_{i+1/2}^{n-1/2}| + \frac{\Delta t P'(\xi_i)}{\mu \Delta x^2} |H_{i-1/2}^{n-1/2}| \leq \left(\frac{\Delta t P'(\xi_i)}{\mu \Delta x^2} + \left|1 - \frac{2 \Delta t P'(\xi_i)}{\mu \Delta x^2}\right| + \frac{\Delta t P'(\xi_i)}{\mu \Delta x^2}\right) \max_i |H_{i+1/2}^{n-1/2}|$$

If

$$0 \leq 1 - \frac{2 \Delta t P'(\xi_i)}{\mu \Delta x^2}, \quad \forall i,$$

we then have

$$|H_{i+1/2}^{n+1/2}| \leq \max_i |H_{i+1/2}^{n-1/2}|$$

and particularly

$$\max_i |H_{i+1/2}^{n+1/2}| \leq \max_i |H_{i+1/2}^{n-1/2}|.$$
From condition (40), we obtain the time step criterion
\[ \Delta t \leq \frac{\mu \Delta x^2}{2 P'_{\text{max}}}, \]
where \( P'_{\text{max}} = \max_i P'(J_i^{n-1/2}) \).

For an electrical resistivity modeled using power law (4), we have
\[ P'(J) = \rho(|J|) + \rho'(|J|) |J|. \]

Considering the superconducting state, the current density should be such that
\[ \frac{|J|}{J_c} \leq C, \]
where \( C \) is a positive constant close to 1, and therefore
\[ \max_{x,t} P'(J) \leq \frac{E_c}{J_c} (p-1) C^{p-2}. \]

If \( \frac{|J|}{J_c} > C \), the superconducting state is lost and the material will behave as a non-superconducting material. The time step condition given by Theorem 4.2 is also similar to the one obtained by Sykulski et al. for the \( E \) formulation [52]. We also note that a similar time step criterion is found in Jin’s paper [23] where he dealt with relaxation models with hyperbolic limits.

4.4. Comparison to Yee’s Scheme

In this section, Scheme A is briefly compared to Yee’s scheme. The relaxation model adds an auxiliary variable, called \( E \), to the original \( p \)-curl problem (3), yet there is no a priori reason to believe that this auxiliary variable should represent a physical quantity such as the electric field, that is besides an analogy to Ohm’s law (2). Nevertheless, the numerical discretization itself takes advantage of this analogy by introducing a staggered scheme with a treatment of \( E \) similar to the one done in Yee’s Scheme for the electric field [51]. It is therefore natural to ask whether we could have constructed Scheme A starting only from Yee’s scheme, and in particular, without the relaxation approximation.

Let’s consider 1-D Maxwell’s equations with \( B = \mu H, D = \epsilon_e E \) and \( J = \sigma(E) E \) here given by
\[ \begin{aligned} \mu \partial_t H - \partial_x E &= 0, \\
\epsilon_e \partial_t E - \partial_x H &= -\sigma(E) E. \end{aligned} \quad (41) \]

For HTS, the term \( \sigma(E) E \) is given by the continuous inverse of \( \rho(J) J = E \),
\[ \sigma(E) E = \begin{cases} \frac{J}{E^{1/p}} |E|^{-\frac{p+1}{p}} E & \text{if } E \neq 0, \\
0 & \text{otherwise}. \end{cases} \quad (42) \]
Yee’s scheme applied on system (41) with the nonlinear conductivity (42) leads to

\[ \mu \frac{H_{i+1/2}^{n+1} - H_{i+1/2}^{n-1/2}}{\Delta t} = E_{i+1}^n - E_i^n, \]

\[ \epsilon_c \frac{E_{i+1}^{n+1} - E_i^n}{\Delta t} - \frac{H_{i+1/2}^{n+1} - H_{i-1/2}^{n+1}}{\Delta x} = - \frac{J_c}{E_{E_c}^{1/p}} \left| \frac{E_{i+1}^{n+1} + E_i^n}{2} \right|^{p-1} \frac{E_{i+1}^{n+1} + E_i^n}{2}. \]

(43)

This scheme needs an implicit treatment for the nonlinear term \( \sigma(E) E \). Even with a splitting into convective and diffusive terms, it will be impossible to solve explicitly for \( E_{i+1}^n \). Moreover, the linearization of the term \( \sigma(E) E \) must be treated carefully because the nonlinear conductivity \( \sigma(E) \) is infinite for \( E = 0 \).

However, one could exploit the magneto-quasistatic condition to obtain the simplified scheme

\[ \mu \frac{H_{i+1/2}^{n+1} - H_{i+1/2}^{n-1/2}}{\Delta t} = E_{i+1}^n - E_i^n, \]

\[ H_{i+1/2}^{n+1} - H_{i-1/2}^{n+1} = J_c \frac{E_{E_c}^{1/p}}{2} \left| \frac{E_{i+1}^{n+1} + E_i^n}{2} \right|^{p-1} \frac{E_{i+1}^{n+1} + E_i^n}{2}. \]

(44)

The second equation of scheme (44) corresponds to a discrete version of Ohm’s law and therefore the electrical resistivity can be used to avoid the linearization of the term \( \sigma(E) E \), which leads to

\[ \mu \frac{H_{i+1/2}^{n+1} - H_{i+1/2}^{n-1/2}}{\Delta t} = E_{i+1}^n - E_i^n, \]

\[ \frac{E_{i+1}^{n+1} + E_i^n}{2} = \frac{\rho_i^{n+1/2}}{\Delta x} \left( H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2} \right). \]

(45)

Yee’s scheme (44) is therefore equivalent to the relaxation limit of scheme (32). However, it can be shown that scheme (45) is linearly unstable using a Von Neumann stability analysis.

In conclusion, Yee’s scheme applied directly on system (41) with the nonlinear conductivity leads to the unstable scheme (45). In this sense, the introduction of the relaxation model helps to identify a new class of schemes hidden from the standard ansatz used in the construction of Yee’s scheme.

4.5. A Second-Order Scheme

In Section 4.2, we have shown that there is a first-order error in time coming from the second equation of Scheme A, given by equations (33), where the term \( E_{i+1}^{n+1} \) in the right-hand side is not centered at \( t_{n+1/2} \). When constructing scheme (45), we attempted to center the values of \( E \) and \( \rho(J) \) at time \( t_{n+1/2} \) but the result was observed to be unstable. The construction of a second-order scheme in time should therefore examine an approximation of \( E \) and \( \rho(J) \) at
time $t_{n+1}$, in order to improve the approximation of the second equation of the relaxation model.

We begin our description of a second-order in time scheme, which we call Scheme B, assuming that the values of $E$ are known at time $t_n$ and that the values of $H$ are known times $t_{n-1/2}$ and $t_{n-3/2}$. In fact, we only assume that $J$ is known at times $t_{n-1/2}$ and $t_{n-3/2}$ previously computed from $H$ according to

$$J_i^{n-1/2} = \frac{H_i^{n-1/2} - H_i^{n-1/2}}{\Delta x}.$$  

The relaxation Scheme B is

$$\mu \frac{H_i^{n+1/2} - H_i^{n-1/2}}{\Delta t} - \frac{E_i^{n+1} - E_i^n}{\Delta x} = F_i^{n+1/2},$$

$$3 E_i^{n+1} - 4 E_i^n + E_i^{n-1} = -\alpha \frac{\Delta t}{2} J_i^{n+1} = -\frac{1}{\epsilon} \left( E_i^{n+1} - P(J_i^{n+1}) \right),$$  

where

$$J_i^{n+1} = \frac{3}{2} J_i^{n+1/2} - \frac{1}{2} J_i^{n-1/2},$$

is a linear extrapolation in time of the current up to time $t_{n+1}$, and hence second-order accurate. It is clear, based on our earlier truncation error analysis given in Section 4.2 and the use of the second-order backward differentiation formula (BDF) in time of $\partial_t E(x_i, t_{n+1})$, that Scheme B is second-order in time as stated in Theorem 4.3. The associated relaxation limit of Scheme B is given by

$$\mu \frac{H_i^{n+1/2} - H_i^{n-1/2}}{\Delta t} - \frac{E_i^{n+1} - E_i^n}{\Delta x} = F_i^{n+1/2},$$

$$E_i^{n+1} = P(J_i^{n+1}).$$

We also perform a Von Neumann stability analysis in Theorem 4.4 and show that the relaxation limit of Scheme B is linearly stable.

**Theorem 4.3.** For the relaxation Scheme B and its relaxation limit, given respectively by (46) and (48), the truncation error is

$$O(\Delta t^2 + \Delta x^2 + \Delta t^2 \Delta x^2).$$

**Theorem 4.4.** Consider the relaxation limit of Scheme B given by (48) with $F = 0$. If the time step satisfies

$$\Delta t \leq \frac{\mu \Delta x^2}{4 P'_{\max}},$$

where $P'_{\max} = \max_{i,n} P'(J_i^n)$, then the relaxation limit of Scheme B is linearly stable.
Proof. We combine the two equations of (48). This leads to

\[ H_{i+1/2}^{n+1} = H_{i+1/2}^{n-1} + \frac{\Delta t}{\mu \Delta x} \left( P(\tilde{J}_{i+1}^n) - P(\tilde{J}_i^n) \right). \]  

(49)

Using the mean value theorem in a similar way than in Theorem 4.2, we obtain

\[ H_{i+1/2}^{n+1} = H_{i+1/2}^{n-1/2} + \frac{\Delta t P'(\xi_i)}{2 \mu \Delta x^2} \left( H_{i+1/2}^{n-3/2} - 3 H_{i+1/2}^{n-3/2} + \frac{3}{2} H_{i+1/2}^{n-3/2} + \frac{1}{2} H_{i+1/2}^{n-3/2} \right) \]

(50)

for \( \xi_i \) strictly between \( \tilde{J}_{i+1}^n \) and \( \tilde{J}_i^n \). Let us consider a single wave number \( k \) and \( H_{i+1/2}^{n-3/2} = e^{i k (i+1/2) \Delta x} \) where \( i \) is the imaginary number. We then have

\[ H_{i+1/2}^{n-1/2} = g(k) e^{i k (i+1/2) \Delta x} \quad \text{and} \quad H_{i+1/2}^{n+1/2} = g^2(k) e^{i k (i+1/2) \Delta x}, \]

where \( g(k) \) is the growth factor. Equation (50) can be written as

\[ g^2(k) - (1 + 3 \gamma) g(k) + \gamma = 0, \]

(51)

where \( \gamma = \frac{\Delta t P'(\xi_i)}{2 \mu \Delta x^2} (\cos(k \Delta x) - 1) \). Solving problem (51) and imposing \( |g(k)| \leq 1 \) to have a stable finite difference scheme, we obtain

\[ -\frac{1}{2} \leq \gamma \leq 0 \]

and

\[ \Delta t \leq \frac{\mu \Delta x^2}{4 P_{\max}'}, \]

(52)

where \( P_{\max}' = \max_{i,n} P'(\tilde{J}_i^n) \). \( \square \)

### 5. 2-D Staggered Discretization

The purpose of this section is to present the extension to 2-D of the proposed Schemes A and B, both for \( \epsilon > 0 \) and for \( \epsilon = 0 \). The presentation will not include analysis of its truncation error, which in any case would be almost identical to the work done in Section 4.

Consider the rectangular cross-section of a superconducting wire of infinite length. The electric field is given by \( E(x, t) = (0, 0, E(x, y, t)) \) and the magnetic field by \( H(x, t) = (H_x(x, y, t), H_y(x, y, t), 0) \) where \( (x, y) \in \Omega \) are the coordinates in the cross-section. In 2-D, the divergence free condition is not trivially
satisfied and will be verified in Section 5.2. The relaxation model is given by

\[
\begin{align*}
\partial_t (\mu H_x) + \partial_y E &= F_x, \\
\partial_t (\mu H_y) - \partial_x E &= F_y, \\
\partial_t E - a^2 (\partial_x H_y - \partial_y H_x) &= -\frac{1}{\epsilon} \left( E - \rho (|\partial_x H_y - \partial_y H_x|) (\partial_x H_y - \partial_y H_x) \right),
\end{align*}
\]

where \( F = (F_x, F_y)^T \) is the source term satisfying \( \nabla \cdot F = 0 \).

5.1. Numerical Scheme

Consider a rectangular domain \( \Omega = [a_L, a_R] \times [b_L, b_R] \) subdivided into \( N_x \times N_y \) square cells \( \Omega_{i,j} \) of area \( \Delta x \Delta y \) with \( \Delta x := (a_R - a_L)/N_x \) and \( \Delta y := (b_R - b_L)/N_y \). We define the nodes of the grid \( x_{i+1/2} := a_L + i \Delta x \) for \( i = 0, 1, \ldots, N_x \) and \( y_{j+1/2} := b_L + j \Delta y \) for \( j = 0, 1, \ldots, N_y \), which then implies that \( \Omega_{i,j} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}] \). The cell center of \( \Omega_{i,j} \) is given by

\[
(x_i, y_j) = (a_L + (i - \frac{1}{2}) \Delta x, b_L + (j - \frac{1}{2}) \Delta y)
\]

for \( i = 1, \ldots, N_x \) and for \( j = 1, \ldots, N_y \). The variables are staggered in both space and time. The components of the magnetic field are defined at the cell edges at times \( t_{n+1/2} \) and the electric field is defined at the cell centers at times \( t_n \). The analogue in 2-D of Scheme A is the following discretization

\[
\begin{align*}
\mu \frac{H_{x,i,j}^{n+1/2} - H_{x,i,j}^{n-1/2}}{\Delta t} &= -\frac{E_{i,j+1}^n - E_{i,j}^n}{\Delta y} + F_{x,i,j+1/2}^n, \\
\mu \frac{H_{y,i+1/2,j}^{n+1/2} - H_{y,i+1/2,j}^{n-1/2}}{\Delta t} &= \frac{E_{i+1,j}^n - E_{i,j}^n}{\Delta x} + F_{y,i+1/2,j}^n, \\
\frac{E_{i,j+1}^{n+1} - E_{i,j}^{n+1}}{\Delta t} - a^2 J_{i,j}^{n+1/2} &= -\frac{1}{\epsilon} \left( E_{i,j+1}^{n+1} - \rho_{i,j}^{n+1/2} J_{i,j}^{n+1/2} \right),
\end{align*}
\]

with

\[
J_{i,j}^{n+1/2} = \frac{H_{y,i+1/2,j}^{n+1/2} - H_{y,i-1/2,j}^{n+1/2}}{\Delta x} - \frac{H_{x,i+1/2,j}^{n+1/2} - H_{x,i-1/2,j}^{n+1/2}}{\Delta y}
\]

and

\[
\rho_{i,j}^{n+1/2} = \alpha |J_{i,j}^{n+1/2}|^{p-2}.
\]

Henceforth, both (33) and (54) will be referred to as Scheme A. To ensure that the relaxation limit, when \( \epsilon \) tends to zero, is consistent with the \( p \)-curl problem, we define the relaxation limit of Scheme A as

\[
\begin{align*}
H_{x,i,j+1/2}^{n+1/2} &= H_{x,i,j+1/2}^{n-1/2} - \frac{\Delta t}{\mu} \frac{E_{i,j+1}^n - E_{i,j}^n}{\Delta y} + \frac{\Delta t}{\mu} F_{x,i,j+1/2}^n, \\
H_{y,i+1/2,j}^{n+1/2} &= H_{y,i+1/2,j}^{n-1/2} + \frac{\Delta t}{\mu} \frac{E_{i+1,j}^n - E_{i,j}^n}{\Delta x} + \frac{\Delta t}{\mu} F_{y,i+1/2,j}^n, \\
E_{i,j}^{n+1} &= \rho_{i,j}^{n+1/2} J_{i,j}^{n+1/2}.
\end{align*}
\]
Schemes \([54]\) and \([55]\) are only first-order in time, but second-order in space. As in the 1-D case, we introduce a correction term on \(J_{i,j}^{n+1}\) to recover a second-order scheme in both time and space, which will be called Scheme B. The relaxation scheme with the correction is then given by

\[
\begin{align*}
\frac{H_{x,i,j+1/2}^{n+1/2} - H_{x,i,j+1/2}^{n-1/2}}{\Delta t} &= - \frac{E_{i,j+1}^{n} - E_{i,j}^{n}}{\Delta y} + F_{x,i,j+1/2}^{n}, \\
\frac{H_{y,i+1/2,j}^{n+1/2} - H_{y,i+1/2,j}^{n-1/2}}{\Delta t} &= \frac{E_{i+1,j}^{n} - E_{i,j}^{n}}{\Delta x} + F_{y,i+1/2,j}^{n}, \\
\frac{3E_{i,j}^{n+1} - 4E_{i,j}^{n} + E_{i,j}^{n-1}}{2\Delta t} - a^2\tilde{J}_{i,j}^{n+1} &= - \frac{1}{\epsilon} \left( E_{i,j}^{n+1} - P(\tilde{J}_{i,j}^{n+1}) \right),
\end{align*}
\]

with

\[
\begin{align*}
\tilde{J}_{i,j}^{n+1} &= \frac{3}{2} J_{i,j}^{n+1/2} - \frac{1}{2} J_{i,j}^{n-1/2}, \\
J_{i,j}^{n+1/2} &= \frac{H_{y,i+1/2,j}^{n+1/2} - H_{y,i+1/2,j}^{n-1/2}}{\Delta x} - \frac{H_{x,i+1/2,j}^{n+1/2} - H_{x,i+1/2,j}^{n-1/2}}{\Delta y}.
\end{align*}
\]

The associated scheme obtained as a relaxation limit is

\[
\begin{align*}
\frac{H_{x,i,j+1/2}^{n+1/2} - H_{x,i,j+1/2}^{n-1/2}}{\Delta t} &= - \frac{E_{i,j+1}^{n} - E_{i,j}^{n}}{\Delta y} + F_{x,i,j+1/2}^{n}, \\
\frac{H_{y,i+1/2,j}^{n+1/2} - H_{y,i+1/2,j}^{n-1/2}}{\Delta t} &= \frac{E_{i+1,j}^{n} - E_{i,j}^{n}}{\Delta x} + F_{y,i+1/2,j}^{n}, \\
E_{i,j}^{n+1} &= P(\tilde{J}_{i,j}^{n+1}).
\end{align*}
\]

The initial conditions are treated as in the 1-D case, and so the magnetic field is imposed along the boundary.

### 5.2. Discrete Divergence Free Condition

In 2-D, an often-used \([42]\) discrete approximation of the divergence of a vector field \(A = (A_x, A_y, 0)\) is the following quantity

\[
(\nabla \cdot A)_{i+1/2,j+1/2}^{n+1/2} := \frac{A_{x,i+1/2,j+1/2}^{n+1/2} - A_{x,i+1/2,j+1/2}^{n-1/2}}{\Delta x} + \frac{A_{y,i+1/2,j+1}^{n+1/2} - A_{y,i+1/2,j+1}^{n-1/2}}{\Delta y}
\]

which is second-order at the cell corners. Noticing first that the electric field \(E(x, t) = (0, 0, E(x, y, t))\) already satisfies the divergence free constraint, we therefore focus only on the magnetic field. The following lemma shows that if the initial data is discrete divergence free, then the solution will be discrete divergence free at all later times.

**Lemma 5.1.** Assume that the magnetic permeability \(\mu\) is constant and that the initial conditions and source terms satisfy

\[
(\nabla \cdot H)_{i+1/2,j+1/2}^{1/2} = 0, \quad (\nabla \cdot F)_{i+1/2,j+1/2}^{n} = 0,
\]

22
for all \( n \geq 0 \) and all \( i, j \) corresponding to nodes inside \( \Omega \). Then for Schemes A and B, with \( \epsilon > 0 \) or \( \epsilon \rightarrow 0 \), the magnetic field remains discrete divergence free at all times.

**Proof.** This lemma will be demonstrated by induction and will be similar to the proofs found in [42]. At a given time \( t_{n-1/2} \), assume that

\[
(\nabla \cdot H)_{i+1/2,j+1/2}^{n-1/2} = 0, \quad (\nabla \cdot F)_{i+1/2,j+1/2}^{n} = 0, \quad \forall i, j, n.
\]

Apply the discrete divergence operator to the first two equations to scheme (54) and find

\[
(\nabla \cdot H)_{i+1/2,j+1/2}^{n+1/2} = (\nabla \cdot H)_{i+1/2,j+1/2}^{n-1/2} + \frac{\Delta t}{\mu} (\nabla \cdot A)_{i+1/2,j+1/2}^{n} + \frac{\Delta t}{\mu} (\nabla \cdot F)_{i+1/2,j+1/2}^{n},
\]

where \( A \) is an approximation of \( \nabla \times E \) at the cell edges given by

\[
A_{x,i,j+1/2}^{n} = -\frac{E_{i,j+1}^{n} - E_{i,j}^{n}}{\Delta y}, \quad A_{y,i+1/2,j}^{n} = \frac{E_{i+1,j}^{n} - E_{i,j}^{n}}{\Delta x}.
\]

Using \( (\nabla \cdot F)_{i+1/2,j+1/2}^{n} = 0 \) and after a straightforward verification that

\[
(\nabla \cdot A)_{i+1/2,j+1/2}^{n} = 0,
\]

then it follows immediately that \( (\nabla \cdot H)_{i+1/2,j+1/2}^{n+1/2} = 0 \). This proves that the approximation of the magnetic field satisfies \( (\nabla \cdot H)_{i+1/2,j+1/2}^{n+1/2} = 0 \) holds for all \( n \).

**5.3. Stability Analysis**

In this section, we demonstrate a time step criterion which ensures that the relaxation limit of Scheme A (54) is stable in 2-D using a maximum norm stability analysis. For the relaxation limit of the second-order Scheme B (56), we perform a Von Neumann stability analysis as in 1-D. Numerical experiments, not included in this report, indicate that both the stability criterion for the relaxation limit of Scheme A in 1-D and in 2-D (see Theorem 4.2 and Theorem 5.2) are optimal. However, numerical experiments show that the time step criteria (52) in 1-D and (64) in 2-D seem suboptimal for the relaxation limit of Scheme B.

**Theorem 5.2.** Consider the 2-D relaxation approximation (53) of the p-curl with the power-law \( P(J) = \rho(J) J \) with \( p \geq 2 \). Assume that the discrete magnetic field satisfies the assumptions of Theorem 5.1 if the time step satisfies

\[
\Delta t \leq \frac{\mu}{2P'_{\text{max}}} \left( \frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2} \right),
\]

23
where \( P'_{\text{max}} = \max_{i,j} P'(J_{i,j}^{n-1/2}) \), then the relaxation limit of Scheme A is stable in the sense that
\[
\|H^{n+1/2}\|_\infty \leq \|H^{n-1/2}\|_\infty.
\]
In other words, scheme (55) is stable in \( \ell^\infty \).

**Proof.** Consider the relaxation limit of Scheme A given by (55) with \( F = 0 \). Substitute the third equation of scheme (55) in the first two equations. This leads to
\[
\begin{align*}
H^{n+1/2}_{x,i,j+1/2} &= H^{n-1/2}_{x,i,j+1/2} - \frac{\Delta t}{\mu \Delta y} \left( P(J^{n-1/2}_{i,j+1}) - P(J^{n-1/2}_{i,j}) \right), \\
H^{n+1/2}_{y,i+1/2,j} &= H^{n-1/2}_{y,i+1/2,j} + \frac{\Delta t}{\mu \Delta x} \left( P(J^{n-1/2}_{i+1,j}) - P(J^{n-1/2}_{i,j}) \right). 
\end{align*}
\]
(59)

Let us begin with the first equation of (59). By the mean value theorem, we have
\[
P(J^{n-1/2}_{i,j+1}) - P(J^{n-1/2}_{i,j}) = P'_{\xi_{i,j}} (J^{n-1/2}_{i,j+1} - J^{n-1/2}_{i,j}),
\]
for \( \xi_{i,j} \) strictly between \( J^{n-1/2}_{i,j+1} \) and \( J^{n-1/2}_{i,j} \). The magnetic flux is discrete divergence free by Theorem 5.1 hence
\[
\frac{H^{n-1/2}_{y,i+1/2,j+1} - H^{n-1/2}_{y,i+1/2,j}}{\Delta y} = -\frac{H^{n-1/2}_{x,i+1,j+1/2} - H^{n-1/2}_{x,i,j+1/2}}{\Delta x},
\]
which we immediately use to compute
\[
\frac{J^{n-1/2}_{i,j+1} - J^{n-1/2}_{i,j}}{\Delta y} = \frac{H^{n-1/2}_{y,i+1/2,j+1} - H^{n-1/2}_{y,i+1/2,j}}{\Delta y} - \frac{H^{n-1/2}_{y,i+1/2,j}}{\Delta y} - \frac{2 H^{n-1/2}_{x,i,j+1/2} + H^{n-1/2}_{x,i,j-1/2}}{\Delta y^2} = \frac{H^{n-1/2}_{x,i+1,j+1/2} - H^{n-1/2}_{x,i-1,j+1/2}}{\Delta x^2}.
\]
We therefore obtain
\[
|H^{n+1/2}_{x,i,j+1/2}| \leq \frac{\Delta t P^\prime(\xi)}{\mu \Delta y^2} |H^{n-1/2}_{x,i,j+3/2}| + \frac{\Delta t P^\prime(\xi)}{\mu \Delta x^2} |H^{n-1/2}_{x,i,j-1/2}| + \frac{\Delta t P^\prime(\xi)}{\mu \Delta x^2} |H^{n-1/2}_{x,i+1,j+1/2}| + \frac{\Delta t P^\prime(\xi)}{\mu \Delta y^2} |H^{n-1/2}_{x,i,j+1/2}| \leq \frac{2 \Delta t P^\prime(\xi)}{\mu \Delta y^2} + \frac{2 \Delta t P^\prime(\xi)}{\mu \Delta x^2} + \frac{\Delta t P^\prime(\xi)}{\mu \Delta x^2} \max_{i,j} |H^{n+1/2}_{x,i,j+1/2}|, 
\]
where \( \xi = \xi_{i,j} \). If
\[
0 \leq 1 - \frac{2 \Delta t P^\prime(\xi_{i,j})}{\mu \Delta x^2} - \frac{2 \Delta t P^\prime(\xi_{i,j})}{\mu \Delta y^2}, \quad \forall i,j,
\]
(60)
we then have

$$\max_{i,j} |H_{x,i,j+1/2}^{n+1/2}| \leq \max_{i,j} |H_{x,i,j+1/2}^{n-1/2}|.$$  

From (60), we obtain the time step criterion

$$\Delta t \leq \frac{\mu}{2P_{\text{max}}^{i,j}} \left( \frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2} \right), \quad (61)$$

where $P_{\text{max}}^{i,j} = \max_{i,j} P'(J_{i,j}^{n-1/2})$. Following a similar procedure for the second equation of (59) leads to the same time step criterion (61).

**Theorem 5.3.** Consider the relaxation limit of Scheme B given by (57) with $F = 0$. Assume that the discrete magnetic field satisfies the assumptions of Theorem 5.1, if the time step satisfies

$$\Delta t \leq \frac{\mu}{4P_{\text{max}}^{i,j,n}} \left( \frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2} \right),$$

where $P_{\text{max}}^{i,j,n} = \max_{i,j,n} P'(J_{i,j}^{n+1/2})$, then the relaxation limit of Scheme B is linearly stable.

**Proof.** Consider the relaxation limit of Scheme B given by (57) with $F = 0$. Substitute the third equation of scheme (57) in the first two equations. Using the magnetic flux is discrete divergence free by Theorem 5.1, as in Theorem 5.2 to decouple the two first equations of the resulting system and focusing on the $H_x$ component, we obtain

$$H_{x,i,j+1/2}^{n+1/2} = H_{x,i,j+1/2}^{n-1/2} + \frac{3\Delta t P'(\xi_{i,j})}{2\mu \Delta x^2} \left( H_{x,i,j+1/2}^{n-1/2} - 2H_{x,i,j+1/2}^{n-3/2} + H_{x,i,j+1/2}^{n-1/2} \right)$$

$$- \frac{\Delta t P'(\xi_{i,j})}{2\mu \Delta x^2} \left( H_{x,i,j+3/2}^{n-3/2} - 2H_{x,i,j+1/2}^{n-3/2} + H_{x,i,j-1/2}^{n-3/2} \right)$$

$$+ \frac{3\Delta t P'(\xi_{i,j})}{2\mu \Delta x^2} \left( H_{x,i,j+3/2}^{n-1/2} - 2H_{x,i,j+1/2}^{n-1/2} + H_{x,i,j-1/2}^{n-1/2} \right)$$

$$- \frac{\Delta t P'(\xi_{i,j})}{2\mu \Delta x^2} \left( H_{x,i,j+3/2}^{n-3/2} - 2H_{x,i,j+1/2}^{n-3/2} + H_{x,i,j-1/2}^{n-3/2} \right)$$

(62)

for $\xi_{i,j}$ strictly between $\bar{J}_{i,j}$ and $\bar{J}_{i,j+1}$. Following the same steps as in the proof of Theorem 4.4, but taking two different wave numbers $\ell$ and $k$ in the directions of the $x$ and $y$ axis, $H_{x,i,j+1/2}^{n-3/2} = e^{i\ell x} e^{ik(j+1/2)} \Delta y$, $H_{x,i,j+1/2}^{n-1/2} = g(k) e^{i\ell x} e^{ik(j+1/2)} \Delta y$, $H_{x,i,j+1/2}^{n+1/2} = g^2(k) e^{i\ell x} e^{ik(j+1/2)} \Delta y$ in equation (62) gives

$$g^2(k) - (1 + 3\gamma)g(k) + \gamma = 0,$$

(63)

where

$$\gamma = \frac{\Delta t P'(\xi_{i,j})}{\mu} \left( \frac{\cos(\ell \Delta x) - 1}{\Delta x^2} + \frac{\cos(k \Delta y) - 1}{\Delta y^2} \right).$$

25
Hence, the time step condition is given by

$$\Delta t \leq \frac{\mu}{4 P'_{\text{max}}} \left( \frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2} \right),$$

(64)

where $P'_{\text{max}} = \max_{i,j,n} P'(J_{i,j}^{n-1/2})$. Following the same procedure for the $H_y$ component leads to the same time step condition.

6. Numerical Results

In this section, we present numerical results confirming the earlier stability results. We analyze numerical results for 1-D and 2-D problems containing a moving front. Many of the results will be obtained for manufactured solutions, that is problems for which analytic solutions exist, and the exact error and order of convergence can be computed.

6.1. Mayergoyz’s Solution

A simple analytic solution with a moving front is Mayergoyz’s analytic solution [53], which is constructed for the $p$-curl problem in 1-D, but reduces to the $p$-Laplacian. The domain is $\Omega = [0, 1]$ and the solution is computed until time $0.5$. The solution is

$$H(x, t) = \begin{cases} 
C t^{p-1} (1 - \frac{2 x}{\xi_0 t})^{\frac{1}{p-2}} & \text{if } 0 \leq x \leq x_f(t), \\
0 & \text{if } x > x_f(t),
\end{cases}$$

(65)

where $C$ is a constant,

$$\xi_0 = \left( \frac{p - 1}{p - 2} \right)^{1 - \frac{1}{p}}, \quad \gamma = C^{\frac{1}{p-1}} \alpha^{-\frac{1}{p}}, \quad \alpha = \frac{E_\epsilon}{\mu_0 J_c^{p-1}},$$

and the position of the front is given by $x_f(t) = \frac{\xi_0}{\gamma} t$. The parameters of the analytic solution are chosen as $C = 1$, $E_c = 1$, $J_c = 1$ and $\mu = 1$.

Convergence is studied for $\Delta x \in \{0.04, 0.02, 0.0133, 0.01, 0.005, 0.004\}$ and $p \in \{3, 5, 8, 10, 20, 50\}$. The time step is given by $\Delta t = \mu \Delta x^2/(2 P'_{\text{max}})$ for all schemes, except for scheme (48) where we need to use $\Delta t = \mu \Delta x^2/(3 P'_{\text{max}})$ for $p \leq 10$. As we noted earlier, the time step condition for Scheme B in Theorem 4.4 is not optimal. The error is computed using the $L_1$-norm. Graphs of the solution are presented in Figure [1]. Table [1] presents numerical estimates of the orders of convergence for the relaxation schemes with $\epsilon = \Delta x^2$, and their relaxation limit when $\epsilon \to 0$. For the convergence analysis of the exponent $p = 50$, the mesh grid size includes the values listed above, as well as three smaller values until $\Delta x = 1/600$ is reached. The orders of convergence of the magnetic field and the electric field are quadratic for $\epsilon = \Delta x^2$ because the first-order term in $\epsilon$ in the $H$-expansion has the same order with respect to $\Delta x$ as the discretization error given in Theorem [4.1]. For the current density,
Figure 1: Plots of $H$, $E$ and $J$ obtained from the relaxation limit of Scheme A with $\Delta x = 0.004$ applied to Mayergoyz’s solution [65].

Figure 2 presents the error as a function of $\Delta x$ for different values of $\epsilon$ and a demanding value of $p = 50$. Figure 1 shows the solution for different exponents with $\Delta x = 0.004$. The numerical solution for $J$ is sharper as the exponent increases and approaches the solution to Bean’s model [5]. The loss of quadratic convergence on the current density can be explained by the difficulty in resolving the strong gradient.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$H$</th>
<th>$E$</th>
<th>$J$</th>
<th>$H$</th>
<th>$E$</th>
<th>$J$</th>
<th>$H$</th>
<th>$E$</th>
<th>$J$</th>
<th>$H$</th>
<th>$E$</th>
<th>$J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2.00</td>
<td>2.00</td>
<td>1.89</td>
<td>2.00</td>
<td>1.98</td>
<td>1.85</td>
<td>2.00</td>
<td>1.99</td>
<td>1.89</td>
<td>2.00</td>
<td>1.98</td>
<td>1.91</td>
</tr>
<tr>
<td>5</td>
<td>2.03</td>
<td>2.01</td>
<td>1.47</td>
<td>2.01</td>
<td>2.06</td>
<td>1.40</td>
<td>2.04</td>
<td>2.03</td>
<td>1.46</td>
<td>2.02</td>
<td>2.07</td>
<td>1.39</td>
</tr>
<tr>
<td>8</td>
<td>2.11</td>
<td>2.17</td>
<td>1.33</td>
<td>2.05</td>
<td>1.98</td>
<td>1.32</td>
<td>2.11</td>
<td>2.16</td>
<td>1.33</td>
<td>1.96</td>
<td>2.03</td>
<td>1.32</td>
</tr>
<tr>
<td>10</td>
<td>1.87</td>
<td>2.18</td>
<td>1.31</td>
<td>1.79</td>
<td>2.14</td>
<td>1.26</td>
<td>1.86</td>
<td>2.18</td>
<td>1.31</td>
<td>1.70</td>
<td>2.14</td>
<td>1.26</td>
</tr>
<tr>
<td>20</td>
<td>2.16</td>
<td>2.15</td>
<td>0.81</td>
<td>2.05</td>
<td>2.02</td>
<td>0.80</td>
<td>2.16</td>
<td>2.15</td>
<td>0.82</td>
<td>2.05</td>
<td>2.01</td>
<td>0.80</td>
</tr>
<tr>
<td>50</td>
<td>1.84</td>
<td>1.64</td>
<td>1.34</td>
<td>1.75</td>
<td>1.70</td>
<td>1.32</td>
<td>1.84</td>
<td>1.64</td>
<td>1.34</td>
<td>1.75</td>
<td>1.69</td>
<td>1.32</td>
</tr>
</tbody>
</table>

Table 1: Estimates of the order of convergence order in $L_1$-norm for relaxation Schemes A and B with $\epsilon = \Delta x^2$, and their relaxation limit ($\epsilon \to 0$) applied to Mayergoyz’s solution [65].
(a) The magnetic field for \( p = 50 \).

(b) The electric field for \( p = 50 \).

(c) The current density for \( p = 50 \).

Figure 2: The error in \( L_1 \)-norm versus \( \Delta x \) for \( p = 50 \) for Mayergoyz’s solution (65) using Scheme A. Similar graphs were used to computed the orders of convergence in Table II for different values of \( p \).
6.2. Manufactured Solution in 2-D

Consider a rectangular domain $[0, 1]^2$ over which we attempt to solve the $p$-curl problem until the time $T = 0.3$ is reached. We set all parameters to one except for the relaxation parameter and the power-law’s exponent. We propose the following manufactured solution

$$H_x(x, y, t) = \frac{1}{4} e^{-x-y-t},$$

$$H_y(x, y, t) = \frac{1}{4} \tan(xy) e^{-t},$$

which is constructed to possess a steep gradient in one of the corners of $\Omega$, particularly in $E$. The time step is given by $\Delta t = \mu h^2/(4 P'_\text{max})$, where $h = \Delta x = \Delta y$, for all schemes, except for scheme (57). As in the 1-D case, we have found in practice that the time step condition in Theorem 5.3 is not optimal and a less restrictive time step of $\Delta t = \mu h^2/(6 P'_\text{max})$ is sufficient. We use $h \in \{0.04, 0.02, 0.01, 0.005, 0.004\}$ and the error is computed using again the $L_1$-norm.

Table 2 shows respectively the convergence plot for $\epsilon = h^2$ and $\epsilon \to 0$. For $\epsilon > 0$, the magnetic field and the electric field have second-order convergence for $\epsilon = h^2$. For the relaxation limit of Schemes A and B, we obtain a second order convergence for all variables. The results suggest that the implementation in 2-D is correct and that it ables to handle fronts adequately.

<table>
<thead>
<tr>
<th>Scheme A [54]</th>
<th>$\epsilon = \Delta x^2$</th>
<th>relaxation limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$H_x$</td>
<td>$H_y$</td>
</tr>
<tr>
<td>3</td>
<td>1.96</td>
<td>1.98</td>
</tr>
<tr>
<td>5</td>
<td>1.98</td>
<td>1.99</td>
</tr>
<tr>
<td>8</td>
<td>1.92</td>
<td>1.93</td>
</tr>
<tr>
<td>10</td>
<td>1.90</td>
<td>1.89</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scheme B [56]</th>
<th>$\epsilon = \Delta x^2$</th>
<th>relaxation limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$H_x$</td>
<td>$H_y$</td>
</tr>
<tr>
<td>3</td>
<td>1.97</td>
<td>1.99</td>
</tr>
<tr>
<td>5</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>8</td>
<td>1.92</td>
<td>1.94</td>
</tr>
<tr>
<td>10</td>
<td>1.89</td>
<td>1.88</td>
</tr>
</tbody>
</table>

Table 2: Estimates of the order of convergence in $L_1$-norm for relaxation Schemes A and B with $\epsilon = h^2$, and their relaxation limit ($\epsilon \to 0$) applied to the manufactured solution (66).
6.3. Focusing Magnetic Front in 2-D

The numerical experiment is intended to roughly model the magnetic field inside a rectangular superconductor initialized to zero but generated by an external field. The objective is to see if the scheme is non-oscillatory. Given the diffusive time step constraint $\Delta t \propto \Delta x^2$, we have found that Schemes A and B provide approximations with almost the same accuracy in practice and we have therefore only used Scheme A for this 2-D experiment.

The geometry is a rectangular domain $\Omega = [0, 1]^2$ and the time interval is $[0, 0.25]$. We impose $H_x = t$ at the top, $H_x = -t$ at the bottom, $H_y = t$ at the left and $H_y = -t$ at the right boundary. All the physical parameters are set to one except for the power-law’s exponent. We initialize all the variables to zero. We use $N_x = N_y = 50$ and the time step $\Delta t = \mu h^2/(4 P_{\text{max}}')$ of Theorem 5.2. In cylindrical coordinates, Bean’s model solution provides an approximately linear evolution on the magnetic field’s $\theta$-component and a Heaviside step function evolution on the current density’s $z$-component [54]. The fields penetrate from the outside towards the inside.

Figure 3 presents the magnetic field and the electric field with $p = 50$ at $t = 0.18$. As expected, the magnetic field has an approximate linear evolution on both components. The evolution of the current density through time is presented for various exponent’s values in Figure 4 and Figure 5. The density current is closer to a Heaviside step function as the exponent increases. Even with $p = 50$, oscillations are absent.

6.4. Magnetization of an HTS wire in 2-D

The purpose of this numerical experiment is to examine closely the magnetic field across a sharp front to confirm that spurious oscillations are suppressed, as expected since we have shown that the scheme satisfies a maximum principle. This numerical experiment is based on laboratory experiments performed by [55] with additional values of the parameters taken from related experiments.

We study the magnetization of a square HTS wire composed of three different materials, namely the barrier, the sheath and the HTS. Figure 6 illustrates a typical geometry of the cross-section of a square HTS wire. The electrical...
Figure 4: The evolution of the current density for $p = 5, 10, 50$ for the magnetic front problem.

Figure 5: The evolution of the current density along $x = 0.51$ for $p = 5, 10, 50$ for magnetic front problem. The dashed black line and the black line are respectively for $t = 0.12$ and $t = 0.25$. 
Figure 6: Geometry of the cross-section of a square HTS wire.

The resistivity model is the same on each subdomain, hence

\[ \rho = \begin{cases} 
\rho_s & \text{if } (x, y) \in \Omega_s, \\
\rho_b & \text{if } (x, y) \in \Omega_b, \\
\alpha |\partial_x H_y - \partial_y H_x|^p & \text{if } (x, y) \in \Omega_{HTS}. 
\end{cases} \]

The HTS wire is exposed to a sinusoidal external magnetic field of the form

\[ H(x, y, t) = (0, B_a \mu_{\text{m}}^{-1} \sin(2\pi ft), 0), \]

where \( \mu = 4\pi \times 10^{-7} \) N/A², \( f = 72 \) Hz is a frequency and \( B_a = 0.0716 \) T is the amplitude of the magnetic induction field. As for the magnetic front, we should observe eddy currents and the fields should penetrate from the outside towards the inside of the wire. We choose a MgB₂-Ti-Cu wire for which the barrier, the sheath and the HTS are respectively composed of titanium (Ti), copper (Cu) and magnesium diboride (MgB₂). The square wire lengths are \( \ell_s = 1.014 \) mm, \( \ell_b = 0.631 \) mm and \( \ell_{HTS} = 0.390 \) mm. The material properties are \( \rho_b = 0.253 \times 10^{-7} \) Ωm, \( \rho_s = 0.830 \times 10^{-10} \) Ωm, \( p = 73 \), \( I_c = 107 \) A and \( J_c = I_c/\ell_{HTS}^2 \) at 30 K [56, 57, 55]. The characteristic lengths are \( H_0 = (4\pi)^{-1} \times 10^7 \) A/m, \( E_0 = 1 \) V/m, \( \rho_0 = \rho_b, \mu_0 = \mu, \ell_0 = 10^{-3} \) m and \( t = 10^{-3} \) s. In the figures, the dimensionless variables are denoted by a tilde ("~"). The mesh grid size is \( h = \Delta x = \Delta y = 1.014/45 \). The time step is given by \( \mu h^2/(4K) \) where \( K = \max\{\rho_b, \rho_s, P'_{\text{max}}\} \).

Figure 7 and Figure 8 illustrate the current density \( J \) at different time steps. We observe that the current density \( J \) penetrates the wire from the outside, as expected. The current density almost vanishes in the barrier, as it should, and eddy currents are generated in the sheath and the HTS. Here again, the proposed numerical methodology provides non-oscillatory approximations, even in the transition between the different materials.
Figure 7: The evolution of the dimensionless current density $\tilde{J}$ for a MgB$_2$-Ti-Cu structure at $B_a = 0.0716$ T, 30 K and 72 Hz for the magnetization of an HTS wire with $h = 1.014/45$. 

(a) $t = \frac{1}{288}$ s
(b) $t = \frac{1}{144}$ s
(c) $t = \frac{3}{288}$ s
(d) $t = \frac{1}{72}$ s
Figure 8: The evolution of the dimensionless current density $\tilde{J}$ for a MgB$_2$-Ti-Cu structure at $B_{a} = 0.0716$ T, 30 K and 72 Hz along $\tilde{y} = 0$ for the magnetization of an HTS wire with $h = 1.014/45$. 
7. Conclusions

This work develops a novel relaxation model for a nonlinear version of the eddy current problem in electromagnetism and suggests several avenues of research. The relaxation model (9) was used to obtain staggered discretization over structured meshes, thus illustrating the effectiveness of the relaxation model in numerical approximation of the $p$-curl. At the moment, the first author is working on a discontinuous Galerkin discretization of the relaxation model in order to maintain the accuracy and non-oscillatory properties of the current scheme, and yet expand its application to more complex geometries [20]. Implementations in 2-D and 3-D of the current staggered scheme, maybe even in cylindrical coordinates, could be used to estimate AC losses in HTS cables or to compute the impedance of strands of twisted HTS wires [40]. Although this work has not emphasized this, the proposed scheme is computationally effective and very competitive with FEM formulations. The predictions from such simulations should be compared to those of more established FE approximations in order to assess the impact of non-oscillatory approximations on the accuracy of quantities of engineering interest.

Acknowledgments

The authors thank Frédéric Sirois for his help on the modelling and applications of superconductors, as well as first introducing the topic to us. We are also grateful to Hongjoong Kim for sharing his insight with us on relaxation models. Ultimately, his point of view had a great influence on how we approached this research.

References


