Mechanisms for error propagation and cancellation in Glimm’s scheme without rarefactions

M. Laforest∗
École Polytechnique de Montréal
département de mathématiques et génie industriel
Montréal, Québec, Canada †
marc.laforest@polymtl.ca

Received (29/09/2006)
Revised (Day Mth. Year)
Communicated by [Philippe G. LeFloch]

Abstract. We derive an a posteriori error bound for Glimm’s approximate solutions to convex scalar conservation laws containing only shock waves. Using Liu’s wave-tracing method, we show that the $L^1$ norm of the error is bounded by a sum of residuals containing independent contributions from each wave in the approximate solution. We introduce a framework, similar to the method of characteristics, for the analysis of the local errors generated by wave interactions. The analysis allows for explicit cancellation among the errors created by a single wave and for error propagation along discontinuities.

Keywords: conservation laws; stability; error estimate.

1. Introduction

We present a new technique to demonstrate the stability in $L^1$ of approximate solutions obtained by Glimm’s scheme for a nonlinear scalar conservation law. The stability estimate holds under the condition that the initial data contain exclusively shock discontinuities. There exists some numerical [10] and mathematical evidence that this restriction can be removed. The technique described here can be interpreted as an extension of Liu’s wave-tracing method [12]. It also provides a bound of the approximation error that is a posteriori, in the sense that it is computable entirely from the approximate solution. Finally, the proof can also be viewed as a more explicit form of a construction of Hoff and Smoller [6].

∗submitted for publication on 29th of September 2006 and supported by National Science and Engineering Council of Canada.
†département de mathématiques et génie industriel, succ. 6079 centre-ville, Montréal, Québec, Canada H3C 3A7
We now briefly summarize the error estimate. Suppose that $u : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is the entropy solution of the scalar nonlinear conservation law

$$u_t + f(u)_x = 0, \quad u(\cdot, 0) = u_0(\cdot) \in L^\infty(\mathbb{R})$$

where $f$ is strictly convex and the initial data $u_0$ is decreasing [2]. Given a sequence of random numbers $\{\theta_k\}_k$, $\theta_k \in [-1, 1]$, Glimm [3] showed how to construct an approximate solution $w$ to the conservation law that converged, for almost all sequences, to the entropy solution as the spatial discretization $\Delta x \rightarrow 0$. Liu [12] later showed how to reinterpret the solution $w$ as a family of waves $W$, i.e. discontinuities, that travel and interact according to random patterns set by the sequence $\{\theta_k\}_k$. For each wave $\alpha \in W$ and at each timestep $t_k \equiv k\Delta t$, the error locally created by the random displacement of the wave can be measured by an appropriately defined and locally computable residual $R(\alpha, t_k)$. We show that if $u_0(\cdot) - w(\cdot, 0) \in L^1(\mathbb{R})$, then

$$\|u(\cdot, t_n) - w(\cdot, t_n)\|_{L^1(\mathbb{R})} \leq \|u_0(\cdot) - w(\cdot, 0)\|_{L^1(\mathbb{R})} + \sum_{\alpha \in W} \left| \sum_{k=1}^n R(\alpha, t_k) \right|. \quad (1.2)$$

The upper bound is computable entirely from $w$ without a priori knowledge of $u$. It is free of unknown constants and valid for all sequences of random numbers. In [10], we have presented numerical results that suggest that the bound is close to optimal as $\Delta x \rightarrow 0$, even for initial data containing rarefactions.

The techniques presented in this proof, if ever extended to allow rarefaction waves in the initial data, might be useful to treat problems requiring an analysis in BV$_{\text{local}}$ such as when the initial data is periodic or in $L^\infty$ [4]. The approach builds upon the method of characteristics and appears to be compatible with the existence and decay theory of Glimm and Lax for 2x2 systems [4]. As an a posteriori error estimate, it is also of interest since it shows explicitly that the errors are created, propagated, and cancelled at the level of waves. The definition of residuals $R(\alpha, t_k)$ used in (1.2) has been used before to bound the error for front-tracking approximations to solutions of hyperbolic systems of nonlinear conservation laws [9], but in that work, error cancellation was not accounted for. Our definition of residual $R(\alpha, t_k)$ has been shown to be a measure of entropy production [10].

This estimate contrasts with previous error bounds for finite difference schemes, such as those of Kröner and Ohlberger [7], Küther [8], Gosse and Makridakis [5], that are of the form $\sum_\alpha \sum_k |R(\alpha, t_k)|$ and therefore tend to overestimate the error. Moreover, the estimate also contrasts with those works since the upper bound does not possess stability constants that tend to be either difficult to compute or excessively large. Adjoint methods have been applied to obtain error estimates that exhibit error cancellation and propagation for finite element solutions to linear (or linearized) conservation laws, but these techniques have not been used successfully for error estimation for finite differences schemes. For a more complete survey of error estimation for compressible fluid flow, see [1].
The proof introduces a series of abstract shock trajectories \( X^{(k)}(t) \) satisfying for all \( \nu, \eta \), that \( X^{(k)}(0) \) and \( X^{(N)}(t) \) are respectively the positions of the wave in \( w \) and \( v \). The local errors are measured with local \( L^1 \)-like discrepancies \( d(X^{(k)}(t), X^{(0)}(t)) = ||\sigma_\alpha| \cdot |x^{(k)}_\nu(t) - x^{(0)}_\nu(t)||_\alpha \), where \( \sigma_\alpha \) is the strength of \( \alpha \). We show that there exists a sequence of geometric transformations mapping the trajectories \( X^{(k)} \) to \( X^{(k+1)} \) and that for each of these, there exists linear transformations \( C^{(k)}(t) \) satisfying for all \( t \), \( 1^T C^{(k)}(t) = 1^T \) and
\[
d(X^{(k+1)}(t), X^{(0)}(t)) \leq C^{(k)}(t) \cdot d(X^{(k)}(t), X^{(0)}(t)). \tag{1.3}
\]
We then show that \( 1^T d(X^{(0)}(t), X^{(0)}(t)) = \|v(\cdot, t) - w(\cdot, t)\|_{L^1} \) and define \( X^{(1)} \) in such a way that \( d(X^{(1)}(t), X^{(0)}(t)) = || \sum_\alpha R(\alpha, t_n)||_\alpha \). The estimate (1.2) then follows by repeated application of (1.3) combined with these last two identities.

The paper is organized as follows. In Section 2 we review Glimm’s scheme, define the residual of the approximate solution, and consider the errors generated by a single isolated traveling wave. In Section 3, we present the main results and show how certain local estimates are used to demonstrate the global estimate. The proof of the local estimates, and most of the novelty in this approach, are detailed in Section 4. We conclude with a discussion of future avenues of research.

2. Preliminaries

2.1. Glimm’s scheme and Liu’s wave tracing method

We begin by reviewing Glimm’s scheme, although we restrict the presentation to hyperbolic scalar conservation laws. A description of the scheme for hyperbolic systems of conservation laws can be found in either [3] or [15].

Given initial data \( v_0 \in L^\infty \), choose a discretization \( \Delta x, \Delta t \) satisfying the following Courant-Friedrichs-Levy (CFL) condition
\[
\sup_x |f'(v)| \leq \frac{\Delta x}{\Delta t}, \tag{2.1}
\]
where the supremum is taken over the convex hull of the range of \( v_0 \). Suppose that \( w(\cdot, 0) \) is a piecewise constant approximation to \( v_0 \) and that it is constant along the intervals \( [(m-1)\Delta x, (m+1)\Delta x] \) with \( m \) even, and with values bounded above and below by the maximum and minimum of \( v_0 \). We may assume that the difference in \( L^1 \) between \( v_0 \) and \( w(\cdot, 0) \) is bounded. Among the class of solutions in \( v \in L^\infty([0, T], BV(\mathbb{R})) \cup Lip([0, T], L^1(\mathbb{R})) \), there exists a unique weak solution of (1.1) that satisfies, in a weak sense, the inequality
\[
\nu(v) \xi + \eta(v) \sigma \leq 0,
\]
for all pairs \( (\nu, \eta) \) satisfying \( \eta'(u) = f'(u)\nu'(u) \), called entropy-entropy flux pairs [2]. This solution is called the entropy solution. Let \( u \) be the entropy solution with initial data \( u_0(\cdot) = w(\cdot, 0) \). In this report, we will focus on comparing \( w \) to \( u \) rather than directly to \( v \).
A Riemann problem is a Cauchy problem with initial data formed by a single discontinuity separating two constant left and right hand states $u^l$ and $u^r$. For a strictly convex scalar conservation law, the solution of a Riemann problem falls into one of two categories. When $u^l > u^r$, the solution is a discontinuity between the states traveling with velocity given by the Rankine-Hugoniot relation

$$S(u^l, u^r) = \frac{f(u^l) - f(u^r)}{u^l - u^r}. \quad (2.2)$$

When $u^l < u^r$, the solution is called a rarefaction wave and is described as follows. Assuming that $u(\cdot, 0)$ is equal to $u^l$ for $x < 0$ and to $u^r$ for $x \geq 0$, then the rarefaction wave solution is

$$u(x, t) = \begin{cases} 
  u^l & \text{if } x < f'(u^l)t, \\
  r(x/t) & \text{if } x \in [f'(u^l)t, f'(u^r)t], \\
  u^r & \text{if } x > f'(u^r)t,
\end{cases} \quad (2.3)$$

where $r$ is the inverse of $f'$. The approximate solution $w$ given by Glimm’s scheme is piecewise constant as a function of $x$ for each fixed $t = t_n$. We shall write $I_m = [(m - 1)\Delta x, (m + 1)\Delta x)$ and $J_n = [n\Delta t, (n + 1)\Delta t)$. The function $w$ will take constant values over $I_m \times \{t = t_n\}$ if $n + m = 0 \mod 2$. Assuming that the approximate solution $w$ is known at time $t_n$, use $w(\cdot, t_n)$ as initial data for (1.1) along the line $t = t_n$ and let $w$ be equal to the entropy solution inside the strip $\mathbb{R} \times J_n$. The entropy solution inside the strip $\mathbb{R} \times J_n$, because of the CFL condition (2.1), is simply the solution of a sequence of independent Riemann problems of either shock or rarefaction type. In order to make $w(\cdot, t_n)$ into a piecewise constant function along the line $t = t_n + 1$, for $x \in I_{m+1}$, Glimm suggested picking a random number $\theta_{n+1} \in [-1, 1]$ and defining

$$w(x, t_{n+1}) = \lim_{t \to t_{n+1}} w((m + 1)\Delta x + \theta_{n+1}\Delta x, t).$$

This new solution at time $t_{n+1}$ is then piecewise constant and the procedure can be repeated indefinitely to define a piecewise constant solution at all later times. Notice that we have assumed that the random choice $\theta_{n+1}$ is the same for each interval $I_{m+1} \times \{t_{n+1}\}$, although our results continue to hold even if this is not the case.

Liu [11] was the first to observe that the discretization introduced in Glimm’s scheme allows one to reinterpret the approximate solution as a linear superposition of discrete waves propagating and interacting nonlinearly. Liu’s so-called wave-tracing description, as it applies to scalar conservation laws, is summarized in the following Theorem. This version is a restriction to scalar conservation laws of Liu’s more general result for systems of conservation laws.

**Theorem 2.1.** [11] For a positive $\epsilon$, a random sequence $\{\theta_n\}_{n \in \mathbb{N}}$, and a region $\Lambda = \mathbb{R} \times [0, T]$, Glimm’s approximate solution $w$ to a scalar conservation law can be described as a family of waves $\mathcal{W}$, where each wave $\alpha \in \mathcal{W}$ has the following characteristics,
(i) an interaction time \( \tau_\alpha \in [0, T] \), which is an integer multiple of \( \Delta t \),
(ii) two constant left and right hand states \( w^\alpha_l \) and \( w^\alpha_r \),
(iii) a strength
\[
\sigma_\alpha \doteq w^\alpha_r - w^\alpha_l < \epsilon,
\]
(iv) and a position \( x_\alpha(t_n) \in \mathbb{R} \), that is well-defined for \( t_n \leq \tau_\alpha \), and satisfies
\[
x_\alpha(t_{n+1}) - x_\alpha(t_n) = \pm \Delta x.
\]

The approximate solution is then constructed as
\[
w(x, t_n) = w(-\infty, 0) + \sum_{\{\alpha|x_\alpha(t_n) \leq x \land t_n < \tau_\alpha\}} \sigma_\alpha.
\]

Moreover, if \( u_0 \in L^\infty(\mathbb{R}) \) and the sequence \( \{\theta_n\} \) is equidistributed, then as \( \Delta x \to 0 \),
\( w \) converges in \( \text{Lip}([0, T], L^1(\mathbb{R})) \cap L^\infty([0, T], \text{BV}(\mathbb{R})) \) to the unique entropy solution \( v \).

Shock waves are the waves satisfying \( \sigma_\alpha < 0 \) and rarefaction waves are those
which satisfy \( \sigma_\alpha > 0 \). Notice that only rarefaction waves have strengths bounded
in absolute value by \( \epsilon \), according to the third condition. Finally, we remark that
the left and right hand states of a wave need to be distinguished from the following
equally useful quantities
\[
\begin{align*}
w^-_\alpha(t_n) & \doteq \lim_{x \to x_\alpha(t_n)^-} w(x, t_n), \\
w^+_\alpha(t_n) & \doteq \lim_{x \to x_\alpha(t_n)^+} w(x, t_n).
\end{align*}
\]

For the sake of completeness, we give a description of the way in which these waves
can be defined in wave-tracing but we omit the proof of convergence.

**Proof.** Recall that \( r \) is the function satisfying \( \xi = f'(r(\xi)) \) and define \( N = \max\{n|t_n < T\} \). Take all the values of the piecewise constant initial data \( w(\cdot, 0) \), all
the values \( r(\theta_n \Delta x/\Delta t) \) for \( n = 0, \ldots, N \), order these in an increasing fashion and
call them \( \{\bar{u}_k\}_k \). At the cost of possibly adding a finite number of values and renaming
the sequence, assume that the increasing sequence also satisfies \( \bar{u}_{k+1} - \bar{u}_k < \epsilon \).
It should be noted that it is not necessary to include in this sequence values outside
the convex hull of the range of \( v_0 \). Obviously, at any given time \( t_n \), the values taken
on by the piecewise constant function \( w(\cdot, t_n) \) will belong to this sequence.

Fix a discontinuity in the initial data \( w(\cdot, 0) \) with left and right hand states
\( u^l \) and \( u^r \), then for each two consecutive values in the sequence which satisfy
\( [\bar{u}_k, \bar{u}_{k+1}] \subseteq [u^l, u^r] \) we define a wave \( \alpha \in \mathcal{W} \). If \( u^l < u^r \) then the left and right
hand states of this wave will be \( u^\alpha_l = \bar{u}_k \) and \( u^\alpha_r = \bar{u}_{k+1} \). If \( u^l > u^r \) then we
will set \( u^\alpha_r = \bar{u}_k \) and \( u^\alpha_l = \bar{u}_{k+1} \). We define \( x_\alpha(0) \) to be the initial position of the discontinuity in \( w(\cdot, 0) \).

Given two states defining a shock discontinuity at time \( t_n \) and a wave \( \alpha \in \mathcal{W} \)
located at that discontinuity, then
\[
x_\alpha(t_{n+1}) = x_\alpha(t_n) + \Delta x \text{ sign } (S(w^-_\alpha(t_n), w^+_\alpha(t_n)))\Delta t - \theta_{n+1} \Delta x.
\]
Therefore, at time $t_{n+1}$, the wave will again be located at the position of some discontinuity in $w(\cdot, t_{n+1})$. Contrary to rarefactions, we can unambiguously speak of the position of the shock at any time $t \in [t_n, t_{n+1})$, namely by setting

$$x_\alpha(t) = x_\alpha(t_n) + (t - t_n)S(w^-_\alpha(t_n), w^+_\alpha(t_n)).$$

In the case where there are no rarefactions in the initial data, the interaction time of shock waves is infinite and will therefore not play a role in this report.

For rarefaction waves, if $t_n \leq \tau_\alpha$ we can define inductively the position as

$$x_\alpha(t_n) = x_\alpha(t_{n-1}) + \text{sign} \left( f'(w^\alpha_l) \Delta t - \theta_n \Delta x \right) \Delta x.$$

We will be using the convention that $\text{sign}(0) = +1$. We omit the definition of $x_\alpha$ for rarefactions since it is rather complex and will not play a role in this work.

Henceforth, we will restrict our attention uniquely to initial data containing shocks. As we mentioned earlier, we will assume that $u$ is the entropy solution with initial data $w(\cdot, 0)$. Therefore, on any compact subset of $\mathbb{R}$, $u(\cdot, t)$ contains only a finite number of shocks and rarefactions and its solution can be entirely described by the method of characteristics. By restricting the initial data to shocks, the waves of $\mathcal{W}$ can be put into a one-to-one correspondence with the set of discontinuities initially located in $w(\cdot, 0)$.

It is now possible to assign at time $t$ to each shock wave $\alpha \in \mathcal{W}$ a position $y_\alpha(t)$ in the entropy solution $u$ by using the following algorithm. First of all, for each shock wave $\alpha \in \mathcal{W}$, we set $y_\alpha(0) = x_\alpha(0)$. At a later time $t$, the position $y_\alpha(t)$ of the wave is equal to the position of the discontinuity to which $\alpha$ initially belonged. This definition continues to hold even after the discontinuities merge in $u$. We now introduce notation similar to (2.5) for the left and right hand states of the discontinuity to which a wave $\alpha$ belongs

$$w^-_\alpha(t) \doteq \lim_{x \to y_\alpha(t)^-} u(x, t),$$
$$w^+_\alpha(t) \doteq \lim_{x \to y_\alpha(t)^+} u(x, t).$$

The family of waves $\mathcal{W}$ has a natural ordering. A wave $\alpha$ is said to be smaller than a wave $\beta$, written $\alpha < \beta$, if $x_\alpha(0) < x_\beta(0)$, or when equality occurs, if $w^d_\alpha > w^d_\beta$.

The main difference between $u$ and $w$ is therefore not the waves but the positions of the waves and especially the time and the order in which they interact.

### 2.2. Residuals

If $w$ is an approximate solution generated by Glimm’s scheme then the residual is, in the sense of distributions,

$$w_t + f(w)_x.$$  \hfill (2.9)

Since $w$ is a weak solution of the conservation law within the open strip $\mathbb{R} \times (t_n, t_{n+1})$ the residual must be concentrated along the line $t = t_n$. 
Lemma 2.2. In a neighborhood of the line \( t = t_n \), the residual of an approximate solution \( w \) obtained by Glimm’s scheme is

\[
\delta_{t_n}(t) \left( \lim_{s \to t_n-} w(x, s) - w(x, t_n) \right).
\]  

(2.10)

Proof. Let \( B = \{(x, t)| (x - x_0)^2 + (t - t_n) < \rho^2 \} \) be a small open ball through the line \( t = t_n \) and assume that \( w \) is smooth in \( B \setminus \{ t = t_n \} \), except possibly along a piecewise smooth curve with a discontinuity at \( t = t_n \). It is sufficient to consider \( \phi \in C_0^\infty(B) \). Write \( B \setminus \{ t = t_n \} = B_a \cup B_b \) where \( B_a \) and \( B_b \) are the connected regions above and below the line. Moreover, split \( B_a = B_a^- \cup B_a^+ \) and \( B_b = B_b^- \cup B_b^+ \) into the subsets to the left and right of the discontinuity. Now, evaluate the residual in weak form using Stokes’ theorem and (2.9).

\[
- \iint_B w\phi_t + f(w)\phi_x \, dx \wedge dt = \iint_{\partial B^+} w\phi \, dx + f(w)\phi \, dt - \iint_{\partial B^-} f(w)\phi \, dx + f(w)\phi \, dt
\]

Along the discontinuity at time \( t \neq t_n \), that is the set \( \partial B^+ \cap \partial B^- \), \( w \) satisfies the Rankine-Hugoniot condition. Along \( \partial B \), \( \phi \) is zero. The only remaining term is along \( t = t_n \). Let \( \alpha \) and \( \beta \) be respectively the minimum and maximum of the set \( B \cap \{ t = t_n \} \). Then we find

\[
- \iint_B w\phi_t + f(w)\phi_x \, dx \wedge dt = \int_{t \to t_n^-} \lim_{s \to t_n-} (w(x, t) - w(x, t_n))\phi(x, t_n) \, dx.
\]

Consider a discontinuity at \( t = t_{n+1} \) in \( w \) located at \( m\Delta x \). Either this discontinuity is the same as the one previously located at either \((m-1)\Delta x\) or \((m+1)\Delta x\), or it is the result of the interaction of the two discontinuities previously located at \((m-1)\Delta x\) and \((m+1)\Delta x\). In either case, there exists a neighborhood of \( m\Delta x \) of width at most \( 2\Delta x \), say \( N_{m,n+1} \), such that for all \( x \) near the boundary of \( N_{m,n+1} \) we have \( w(x, t_{n+1}) = \lim_{t \to t_{n+1}-} w(x, t) \). Clearly this means that for some \( \delta > 0 \), \( w \) is continuous along \( \partial N_{m,n+1} \times (-\delta, \delta) \).

Given any test function \( \phi \in C_0^\infty(N_{m,n+1} \times (-\delta, \delta)) \) that is identically 1 along \( t = t_{n+1} \), except possibly near the boundary \( \partial N_{m,n+1} \), the weak residual (2.10) satisfies

\[
- \iint_{N_{m,n+1} \times (-\delta, \delta)} w\phi_t + f(w)\phi_x \, dx \wedge dt = \int_{N_{m,n+1}} (w(x, t_{n+1}) - w(x, t_{n+1}-)) \, dx.
\]

(2.11)

Therefore, within this restricted class of test functions, the value of the weak residual is in fact independent of \( \phi \). This suggests the following definition.
Definition 2.3. Given a discontinuity at $m\Delta x$ at time $t_{n+1}$, we define the residual of the discontinuity to be

$$\int_{N_{m,n+1}} w(x,t_{n+1}^-) - w(x,t_{n+1}^+) \, dx.$$  

(2.12)

Lemma 2.4. Assume that $w$ is an approximate solution without rarefaction waves obtained by Glimm’s scheme and that $S = [\alpha(1), \ldots, \alpha(a)]$ is the ordered set of all shock waves in $w$ located at the mesh point $(m\Delta x, (n+1)\Delta t)$, $m + n = 0 \mod 2$. Under these conditions,

$$\int_{N_{m,n+1}} (w(x,t_{n+1}^-) - w(x,t_{n+1}^+)) \, dx$$

$$= \sum_{k=1}^{a} \sigma_{\alpha(k)}(\Delta t \text{sign}(s_k \Delta t - \theta_{n+1} \Delta x) - s_k \Delta t),$$

(2.13)

where $s_k = S(w_{\alpha(k)}^-(t_n), w_{\alpha(k)}^+(t_n))$.

Proof. On the segment $I_m \times \{t = t_{n+1}\} \subset \mathbb{R} \times [0,T]$, the function $w(\cdot,t_{n+1})$ takes on constant values to the left and the right of $x = m\Delta x$ which we denote respectively by $w^l$ and $w^r$.

Suppose that the discontinuity at $(m\Delta x, t_{n+1})$ comes from a discontinuity initially at either $((m - 1)\Delta x, t_n)$ or $(m\Delta x, t_n)$. Then from time $t_n$ to $t_{n+1}$, we saw in (2.6) that the discontinuity in $w$ travels a distance

$$\Delta x \text{sign}(s\Delta t - \theta_{n+1} \Delta x),$$

where $s = S(w^l, w^r)$. The discontinuity in $w(\cdot,t_{n+1}^-)$ is therefore located at

$$s\Delta t + \Delta x (m - \text{sign}(s\Delta t - \theta_{n+1} \Delta x)),

$$

while the discontinuity in $w(\cdot,t_{n+1}^-)$ is located at $x = m\Delta x$. Since $s = s_k$ for all $k$, we find

$$\int_{N_{m,n+1}} w(x,t_{n+1}^-) - w(x,t_{n+1}^-) \, dx$$

$$= (w^r - w^l)(m\Delta x - s\Delta t - \Delta x (m - \text{sign}(s\Delta t - \theta_{n+1} \Delta x)))$$

$$= (w^r - w^l)(\Delta x \text{sign}(s\Delta t - \theta_{n+1} \Delta x) - s\Delta t)$$

$$= \sum_{k=1}^{a} \sigma_{\alpha(k)}(\Delta x \text{sign}(s_k \Delta t - \theta_{n+1} \Delta x) - s_k \Delta t).$$

Suppose that the discontinuity at $(m\Delta x, t_{n+1})$ is formed by the interaction at time $t_{n+1}$ of the two discontinuities previously located at $((m - 1)\Delta x, t_n)$ and $((m + 1)\Delta x, t_n)$. If $w^l$ and $w^r$ are the left and right hand states of the discontinuity in $w(\cdot,t_{n+1})$ at $m\Delta x$, then the discontinuity in $w(\cdot,t_n)$ at $(m - 1)\Delta x$ has neighboring states $w^l$ and $w^m$ while the one at $(m + 1)\Delta x$ has neighboring states $w^m$ and $w^r$. Say these two shocks travel with speeds $s^l \equiv S(w^l, w^m)$ and $s^r \equiv S(w^m, w^r)$ during
the interval \((t_n, t_{n+1})\). Over the set \(N_{m,n+1}\), which must include \((m-1)\Delta x + s^l \Delta t \leq (m+1)\Delta x + s^r \Delta t\), we have
\[
\int_{N_{m,n+1}} [w(x, t_{n+1}^-) - w(x, t_{n+1}^+)] \, dx
\]
\[= (w^m - w^l)(m\Delta x - (m-1)\Delta x - s^l \Delta t) + (w^r - w^m)(m\Delta x - (m+1)\Delta x - s^r \Delta t)
\]
\[= (w^m - w^l)(s^l \Delta t - \Delta x \text{sign}(s^l \Delta t - \theta_{n+1} \Delta x)) + (w^r - w^m)(s^r \Delta t - \Delta x \text{sign}(s^r \Delta t - \theta_{n+1} \Delta x))
\]
\[= \sum_{k=1}^{\alpha} \sigma_{\alpha(k)} (\Delta x \text{sign}(s_k \Delta t - \theta_{n+1} \Delta x) - s_k \Delta t).\]

The identity (2.13) shows that the residual of a discontinuity formed of shock waves decomposes into a sum of contributions from each wave.

**Definition 2.5.** The residual of the wave \(\alpha\) at time \(t_n\) is defined to be
\[R(\alpha, t_n) \doteq \sigma_{\alpha} (\Delta x \text{sign}(s \Delta t - \theta_{n+1} \Delta x) - s \Delta t),\]
where \(s = S(w^l_{\alpha}(t_n^-), w^r_{\alpha}(t_n^-))\).

We observe that the residual of a shock wave can be computed a posteriori, that is using only the approximate solution \(w\), since it depends only on the local states \(w^l_{\alpha}, w^r_{\alpha}\) and \(x_{\alpha}(t_n)\) but not on the global properties of either \(w\) or \(u\).

### 3. Main Result

In this section we present our error estimate and provide all the ingredients of its proof. Namely, we state two lemmas, whose proofs are postponed to Section 4, and use them to demonstrate the error estimate.

**Theorem 3.1.** Consider initial data \(v_0 \in L^\infty(\mathbb{R})\) that is decreasing. Let \(v\) be the entropy solution of the nonlinear scalar conservation law
\[
v_t + f(v)_x = 0, \tag{3.1}
v(\cdot, 0) = v_0(\cdot),
\]
where \(f\) is strictly convex. Suppose the approximation \(w(\cdot, 0)\) to the initial data contains only shocks and \(v_0 - w(\cdot, 0) \in L^1(\mathbb{R})\). Then for any \(\Delta t\) satisfying (2.1), any time \(t\), any sequence \(\{\theta_k\}_k\) of numbers in \([-1, 1]\), and any \(c > 0\) for Liu’s wave tracing method, we have that the approximate solution \(w\) obtained with Glimm’s scheme satisfies
\[
\|v(\cdot, t) - w(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|v_0(\cdot) - w(\cdot, 0)\|_{L^1(\mathbb{R})} + \sum_{\alpha \in W} \sum_{k=1}^{n} R(\alpha, t_k), \tag{3.2}
\]
where \(n = \lfloor t/\Delta t \rfloor \doteq \max\{k|k\Delta t \leq t\} \).
The difficulty of this theorem is that the random errors created at each timestep are $O(\Delta x)$ and can be cumulative. Moreover, the waves may interact in $w$ in an almost arbitrary order and at almost any time. Rather than compare the waves in $w$ with those in $v$, we first compare them to those in the entropy solution $u$ defined by the initial data $w(\cdot, 0)$. The first step is to relate the $L^1$ norm in (3.2) to the positions of the shock waves in $u$ and $w$.

**Definition 3.2.** If the waves in $W$ are ordered $\alpha(1), \ldots, \alpha(N)$, then let $X(t) = \begin{bmatrix} x_{\alpha(1)}(t) & \cdots & x_{\alpha(N)}(t) \end{bmatrix}^T$ be the column vector of the trajectories of waves in $w$. Similarly, let $Y(t) = [y_{\alpha(i)}(t)]_{i=1}^N$ be the trajectories of shock waves in $u$.

**Definition 3.3.** Given two sets of trajectories $Z(t) = [z_{\alpha}(t)]_{\alpha \in W}$ and $\tilde{Z}(t) = [\tilde{z}_{\alpha}(t)]_{\alpha \in W}$, we define the discrepancy as the column vector
\[
d(\tilde{Z}(t), Z(t)) = \begin{bmatrix} |\sigma_\alpha| \cdot |\tilde{z}_{\alpha}(t) - z_{\alpha}(t)| \end{bmatrix}_{\alpha \in W}.
\] (3.3)

The discrepancy of a single wave $\alpha$ will be denoted $d_\alpha(\tilde{Z}(t), Z(t)) = |\sigma_\alpha| \cdot |\tilde{z}_{\alpha}(t) - z_{\alpha}(t)|$.

The discrepancy is a way to quantify in $L^1$ the local error.

**Lemma 3.4.** If $u_0(\cdot) = w(\cdot, 0)$ is decreasing and $1^T = [1, 1, \ldots, 1]$ then for any $t \geq 0$,
\[
\|u(\cdot, t) - w(\cdot, t)\|_{L^1(\mathbb{R})} = 1^T d(Y(t), X(t)).
\] (3.4)

**Proof.** Since $u$ and $w$ are decreasing and their difference is bounded in $L^1$, there exists arbitrarily smooth and close (in $L^1$) perturbations, say $u^\epsilon$ and $w^\epsilon$, that are strictly monotonic decreasing and equal outside a compact set. Let $x_u^\epsilon, x_w^\epsilon$ be the inverses of these perturbations. We then conclude that
\[
\|u(\cdot, t) - w(\cdot, t)\|_{L^1(\mathbb{R})} = \lim_{\epsilon \to 0} \|u^\epsilon(\cdot, t) - w^\epsilon(\cdot, t)\|_{L^1(\mathbb{R})}
= \lim_{\epsilon \to 0} \int_{\text{range}(u)} |x_u^\epsilon(\phi, t) - x_w^\epsilon(\phi, t)| d\phi
= \sum_{\alpha \in W} d_\alpha(Y(t), X(t)).
\]

This result shows that to demonstrate Theorem 3.1, we need to relate $d(Y(t), X(t))$ to the vector of residuals $R(t) = [\sum_{k=1}^n R(\alpha, t_k)]$, where $n = \lfloor t/\Delta t \rfloor$. On the other hand, the residuals are themselves directly related to the discrepancies of another pair of trajectories.

**Definition 3.5.** We define the continuous trajectories $X^{(0)}(t) = [x_{\alpha}(0)]_{\alpha \in W}$ by requiring that the position $x_{\alpha}(0)$ of each wave $\alpha$ satisfy
\[
x_{\alpha}(0) = x_{\alpha}(0), \quad x_{\alpha}^{(0)}(t) = S(w^-_{\alpha}(t), w^+_{\alpha}(t)), \quad \forall t > 0.
\] (3.5)
Lemma 3.6. If \( \alpha \in W \) is a shock wave then for all \( t \geq 0 \)
\[
d_{\alpha}(X^{(0)}(t), X(t)) = \left| \sum_{k=1}^{[t/\Delta t]} R(\alpha, t_k) \right|. \tag{3.6}
\]

Proof. Using the definition of the residual (2.14), the definition of \( X^{(0)} \) given by (3.5), the characterization (2.6) of \( X \), and \( n = [t/\Delta t] \), we can verify that
\[
\left| \sum_{k=1}^{n} R(\alpha, t_k) \right| = \left| \sum_{k=1}^{n} \sigma_{\alpha} \left[ \Delta x \text{sign} \left( S(w_{\alpha}^- (t_k-), w_{\alpha}^+(t_k-)) \Delta t - \theta_k \Delta x \right) 
- S(w_{\alpha}^-(t_k-), w_{\alpha}^+(t_k-)) \Delta t \right] \right|
= |\sigma_{\alpha}| \cdot \left| \sum_{k=1}^{n} S(w_{\alpha}^- (t_k-), w_{\alpha}^+(t_k-)) \Delta t 
- \sum_{k=1}^{n} \Delta x \text{sign} \left( S(w_{\alpha}^- (t_k-), w_{\alpha}^+(t_k-)) \Delta t - \theta_k \Delta x \right) \right|
= |\sigma_{\alpha}| \cdot |x^{(0)}_{\alpha}(t_n) - x_{\alpha}(t_n)|.
\]
Since over the time interval \((t_n, t)\) we have \( \dot{x}^{(0)}_{\alpha} = \dot{x}_{\alpha} \), the last expression is equal to \( |\sigma_{\alpha}| \cdot |x^{(0)}_{\alpha}(t) - x_{\alpha}(t)| \).

The problem is now to find a sequence of corrections that, when applied to \( X^{(0)} \), provide the exact trajectories \( Y \) yet preserve the total quantity of residuals \( \int R(t) = \int d(X^{(0)}(t), X(t)) \). The relationship is simpler to understand if we focus on the discontinuities rather than on the individual waves themselves.

Definition 3.7. An ordered set of waves \( S = [\alpha(1), \ldots, \alpha(n)] \) is said to be consecutive if \( \alpha(i) < \alpha(i+1) \) and there is no \( \beta \in W \setminus S \) such that \( \alpha(i) < \beta < \alpha(i+1) \). Henceforth, all sets of waves will be assumed to be ordered. We will also introduce the concatenation operator \( \land \), as in \( [\alpha(1), \ldots, \alpha(n)] \land [\alpha(n+1), \ldots, \alpha(m)] \equiv [\alpha(1), \ldots, \alpha(m)] \), to emphasize that the order is preserved under unions of sets of waves.

Definition 3.8. Consider a consecutive set of waves \( F = [\alpha(1), \ldots, \alpha(n)] \). The free trajectories of \( F \) are the trajectories \( F(t) = [f_{\alpha(i)}(t)]_{i=1}^{n} \) of the discontinuities in the solution \( V(x, t) \) to the conservation law (3.1) with initial data
\[
V(x, 0) = \begin{cases} 
  w_{\alpha(1)}^i & \text{if } x \leq x_{\alpha(1)}(0), \\
  w_{\alpha(i)}^r & \text{if } x_{\alpha(i)}(0) < x \leq x_{\alpha(i+1)}(0) \text{ and } i \in \{1, \ldots, n-1\}, \\
  w_{\alpha(n)}^r & \text{if } x_{\alpha(n)}(0) < x.
\end{cases} \tag{3.7}
\]
We will say that an interaction occurred in \( F \) at time \( t_* \) if it occurred at time \( t_* \) in the entropy solution \( V \).
Free trajectories are nontrivial examples of some of the virtual trajectories that will need to be considered. Other examples will be restrictions to subsets of trajectories defined for all waves.

**Definition 3.9.** Given a set of trajectories $Z(t) = [z_\alpha(t)]_{\alpha \in \mathcal{W}}$ and a subset $S \subset \mathcal{W}$, we define the column vectors
\[
Z(t) \big|_S = [z_\alpha(t)]_{\alpha \in S},
\]
\[
d(Z(t), X(t)) \big|_S = [d_\alpha(Z(t), X(t))]_{\alpha \in S}.
\]

**Definition 3.10.** We say that a real-valued matrix $C \in \mathcal{M}^{m \times n}(\mathbb{R})$ is conservative if
\[
1^T C = 1^T,
\]
where $1^T = [1, 1, \ldots, 1]$ are vectors of unity of the appropriate sizes.

The proof of Theorem 3.1 proceeds by induction on the sequence of times at which certain types of interactions occur. Before presenting the proof, we describe a construction which defines trajectories used in the statement of the induction hypothesis. Afterwards, we state the induction hypothesis and the last two lemmas required in the proof of estimate (3.2). Section 3 then ends with a complete proof of Theorem 3.1.

At time $t_*$, consider the $m$ discontinuities in $w$ and the subdivision of $\mathcal{W}$ into consecutive sets of waves
\[
\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(m)}
\]
that correspond to each of the $m$ discontinuities in $w$. For each set $\mathcal{F}^{(i)}$, suppose there are $n_i$ discontinuities at time $t = t_*$ in the free trajectories $F^{(i)}(t)$ associated to $\mathcal{F}^{(i)}$, and let
\[
S_{(i,1)}, \ldots, S_{(i,n_i)}
\]
be the subdivision of $\mathcal{F}^{(i)}$ into the consecutive sets of waves that belong to each of these $n_i$ discontinuities. This leads to the subdivision of $\mathcal{W}$ given by
\[
S_{(1,1)}, S_{(1,2)}, \ldots, S_{(1,n_1)},
S_{(2,1)}, S_{(2,2)}, \ldots,
S_{(m,1)}, S_{(m,2)}, \ldots, S_{(m,n_m)}.
\]
Let $F$ be the trajectories defined by
\[
F(t) = \begin{bmatrix} F^{(1)}(t) \\ \vdots \\ F^{(m)}(t) \end{bmatrix},
\]
and construct the trajectories $X^{(1)}$ satisfying
\[
X^{(1)}(t) = F(t), \quad \forall t \in [0, t_*],
\]
\[
\dot{X}^{(1)}(t) = \dot{X}(t), \quad \forall t > t_*.
\]
Lemma 3.12. For our purposes though, we will need to consider subdivisions slightly more general than (3.10). We suppose that at time \( t_* \) there exists a one-parameter family of conservative matrices \( A(t) \) such that for all \( t \)

\[
d(X^{(1)}(t), X(t)) \leq A(t) \cdot d(X^{(0)}(t), X(t)).
\]  (3.16)

The inductive proof will demonstrate that if the induction hypothesis holds at time \( t_* \), then it must also hold at time \( T_* \), and in fact at any time \( t \in [t_*, T_*] \). To accomplish this, we need two tools. The first relates \( X^{(1)} \) to \( X^{(2)} \).

Lemma 3.11. If \( X^{(1)} \) and \( X^{(2)} \) are the trajectories constructed from the subdivision (3.10) at time \( t_* \), then there exists a one-parameter family of conservative matrices \( B(t) \) such that for all \( t \), we have

\[
d(X^{(2)}(t), X(t)) \leq B(t) \cdot d(X^{(1)}(t), X(t)).
\]  (3.17)

The second tool relates the free trajectories \( X^{(2)} \) to the exact trajectories \( Y \). For our purposes though, we will need to consider subdivisions slightly more general than (3.12).

Lemma 3.12. Consider a consecutive subsequence of sets from the subdivision (3.10), say

\[
G^{(1)}, \ldots, G^{(k)}
\]  (3.18)

and let \( G^{(i)}(t) = [g^{(i)}(t)]_{\alpha \in G^{(i)}} \) be their free trajectories. Define \( G = G^{(1)} \wedge \ldots \wedge G^{(k)} \) and let \( G(t) = [g_{\alpha}(t)]_{\alpha \in G} \) be the free trajectories of \( G \). For any time \( T \geq t_* \), construct the trajectories \( \tilde{U}(t) \) and \( \tilde{U}(t) = [u_{\alpha}(t)]_{\alpha \in G} \) according to

\[
\tilde{U}(t) \doteq G(t) \quad \text{when } t \leq T,
\]

\[
\tilde{U}(t) \doteq \dot{X}(t)|_G \quad \text{for } t > T,
\]

and

\[
u_{\alpha}(t) \doteq g^{(i)}_{\alpha}(t) \quad \text{when } t \leq T \text{ and } \alpha \in G^{(i)},
\]

\[
u_{\alpha}(t) \doteq \dot{x}_{\alpha}(t) \quad \text{for } t > T.
\]

Under these conditions, there exists a one-parameter family of conservative matrices \( C(t) \) such that for all \( t \),

\[
d(\tilde{U}(t), X(t)|_G) \leq C(t) \cdot d(U(t), X(t)|_G).
\]  (3.19)
The proofs of Lemmas 3.11 and 3.12 are postponed to Section 4 but we can now proceed with the proof of Theorem 3.1.

**Theorem 3.1.** We can reduce the problem of bounding the difference between $w$ and $v$ to that of bounding the difference between $w$ and $u$, the entropy solution with initial data $u(\cdot,0) = w(\cdot,0)$. We remark that the conditions on $v_0$ and $w(\cdot,0)$ guarantee that $v(\cdot,t) - w(\cdot,t)$ is in $L^1(\mathbb{R})$. Using the fact that the evolution operator for a conservation law is contracting in $L^1[2]$, we find

$$
\|v(\cdot,t) - w(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|v(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})} + \|u(\cdot,t) - w(\cdot,t)\|_{L^1(\mathbb{R})}
\leq \|v_0(\cdot) - w(\cdot,0)\|_{L^1(\mathbb{R})} + \|u(\cdot,t) - w(\cdot,t)\|_{L^1(\mathbb{R})}. \quad (3.20)
$$

It therefore suffices to establish an error bound in terms of the residual for the difference between $u$ and $w$.

We now demonstrate that if the induction hypothesis holds at time $t_*$, then the residuals bound the difference between $w$ and $u$ at all times in $t \in [t_*, T_*]$. If in Lemma 3.12 we set $G = W$, $T = t \in [t_*, T_*]$, and take the subdivision (3.18) of $G$ to be (3.10), then we have the identities $U(t) = X^{(2)}(t)$ and $\bar{U}(t) = Y(t)$. With these definitions, Lemma 3.12 provides a conservative matrix $C(t)$ for which

$$
d(Y(t), X(t)) \leq C(t) \cdot d(X^{(2)}(t), X(t)).
$$

We obtain the desired error estimate by applying in the following order, Lemma 3.4, the previous estimate, Lemma 3.11, the induction hypothesis (3.16), Definition 3.10 of conservative matrices, and Lemma 3.6.

$$
\|u(\cdot,t) - w(\cdot,t)\|_{L^1(\mathbb{R})} = 1^T d(Y(t), X(t))
\leq 1^T C(t) \cdot d(X^{(2)}(t), X(t))
\leq 1^T C(t) B(t) \cdot d(X^{(1)}(t), X(t))
\leq 1^T C(t) B(t) A(t) \cdot d(X^{(0)}(t), X(t))
= \sum_{\alpha \in W} \sum_{k=1}^{\lfloor t/\Delta t \rfloor} R(\alpha, t_k) \quad (3.21)
$$

The error estimate (3.2) now follows from eqs. (3.20) and (3.21). It now suffices to show that the induction hypothesis at time $t_*$ implies the induction hypothesis at time $T_*$. It is trivial to demonstrate the induction hypothesis (3.16) at time $t_* = 0$ because $X^{(1)}(0) = X^{(0)}(0) = X(0)$.

At time $T_* > t_*$, the time of the next interaction in $w$ or among the free trajectories, two cases can occur.

**Case # 1** There exists a free trajectory $F^{(k)}$ in which two or more discontinuities meet at time $T_*$. Generically, we can assume that only two discontinuities in (3.11) interacted at time $T_*$, say $S_{(k,l)}$ and $S_{(k,l+1)}$. Henceforth, the sets and
trajectories defined at time $t_*$, like $S_{(i,j)}$ and $X^{(1)}$, will be distinguished from those at time $T_*$ by a superscript tilde, as in $\tilde{S}_{(i,j)}$ and $\tilde{X}^{(1)}$.

Using the observation that $\tilde{X}^{(1)} = X^{(2)}$, Lemma 3.11 and the induction hypothesis (3.16) provide matrices $B(t)$ and $A(t)$ such that for all $t$, we have

$$d(\tilde{X}^{(1)}(t), X(t)) \leq B(t) \cdot d(X^{(1)}(t), X(t)) \leq B(t)A(t) \cdot d(X^{(0)}(t), X(t)).$$

(3.22)

Setting $\tilde{A}(t) = B(t)A(t)$ and observing that $X^{(0)}(t) = \tilde{X}^{(0)}(t)$ suffices to prove that induction hypothesis holds at time $T_*$.

Case # 2 Two discontinuities in $w$ meet at time $T_*$. Suppose these discontinuities are represented by the sets $\mathcal{F}^{(i)}$ and $\mathcal{F}^{(k+1)}$ of the subdivision (3.10). Assuming no other interactions occur at time $T_*$, the subdivision (3.10) then becomes

$$\mathcal{F}^{(i)} = \mathcal{F}^{(i)}, \quad \text{if } i < k,$$

$$\mathcal{F}^{(i)} = \mathcal{F}^{(i+1)}, \quad \text{if } i > k,$$

and

$$\mathcal{F}^{(k)} = \mathcal{F}^{(k)} \land \mathcal{F}^{(k+1)}.$$ Generically, we may also assume that no interactions occurred at time $T_*$ among the discontinuities of the free trajectories $F^{(i)}, i \neq k, k + 1$, and therefore that the subdivisions (3.11) of $\mathcal{F}^{(i)}$ remain unchanged, that is to say $\tilde{S}_{(i,j)} = S_{(i,j)}$ as long as $i \neq k, k + 1$.

To analyze the free trajectories of $\mathcal{F}^{(k)}$, consider the application of Lemma 3.12 at time $T = T_*$ to the set $\mathcal{G} = \mathcal{F}^{(k)}$ with the subdivision (3.18) given by

$$\mathcal{G}^{(1)} = \mathcal{F}^{(k)}, \quad \mathcal{G}^{(2)} = \mathcal{F}^{(k+1)}.$$ The trajectories in Lemma 3.12 then correspond to

$$U \equiv X^{(2)}\big|_{\mathcal{F}^{(k)}}, \quad \text{and} \quad \tilde{U} \equiv \tilde{X}^{(1)}\big|_{\mathcal{F}^{(k)}},$$

Lemma 3.12 states that there exists a one-parameter family of conservative matrices $C(t)$ such that for all $t$,

$$d(\tilde{X}^{(1)}(t), X(t))\big|_{\mathcal{F}^{(k)}} \leq C(t) \cdot d(X^{(2)}(t), X(t))\big|_{\mathcal{F}^{(k)}}.$$ (3.23)

For the waves in $W \setminus \mathcal{F}^{(k)}$, the subdivision (3.10) is unchanged from the one at time $t_*$, i.e. $\mathcal{F}^{(i)} = \mathcal{F}^{(i)}$ for $i \neq k$, and therefore

$$\tilde{X}^{(1)}\big|_{W \setminus \mathcal{F}^{(k)}} = X^{(2)}\big|_{W \setminus \mathcal{F}^{(k)}},$$

$$d(\tilde{X}^{(1)}(t), X(t))\big|_{W \setminus \mathcal{F}^{(k)}} = d(X^{(2)}(t), X(t))\big|_{W \setminus \mathcal{F}^{(k)}}.$$ (3.24)

Construct a conservative matrix $\tilde{C}(t)$ that is equal to the identity except for the entries belonging to $\mathcal{F}^{(k)} \times \mathcal{F}^{(k)}$ where we set it to $C(t)$. More specifically, take

$$\tilde{C}(t) = \begin{bmatrix} I & 0 & 0 \\ 0 & C(t) & 0 \\ 0 & 0 & I \end{bmatrix}.$$ (3.25)
Combining inequality (3.23), the observation (3.24), Lemma 3.11 and the induction hypothesis (3.16), we find that for all \( t \),
\[
d(\tilde{X}^{(1)}(t), X(t)) \leq \tilde{C}(t) \cdot d(X^{(2)}(t), X(t)) \\
\leq \tilde{C}(t)B(t) \cdot d(X^{(1)}(t), X(t)) \\
\leq \tilde{C}(t)B(t)A(t) \cdot d(X^{(0)}(t), X(t)).
\]
Since \( \tilde{X}^{(0)} = X^{(0)} \), this completes the proof of the induction hypothesis at time \( t = T^* \).

4. Error Cancellation and Interactions

The objective of this section is to prove Lemmas 3.11 and 3.12. These results follow from the principle of conservation of mass as expressed in Lemma 4.4 and its corollaries, Lemmas 4.5 and 4.6. These last two lemmas provide the constructions of the one-parameter family of conservative matrices which are fundamental to Lemmas 3.11 and 3.12. We begin the section with some notation and a few preliminary results.

**Definition 4.1.** Consider a subdivision of a consecutive set of waves \( S \), say \( S_1, \ldots, S_s \). We say that the trajectories \( Z(t) = [z_{\alpha}(t)]_{\alpha \in S} \) are consistent with the subdivision at time \( t^* \) if for each \( i = 1, \ldots, s \), we have
\[
z_{\alpha}(t) = z_{\beta}(t), \quad \forall \alpha, \beta \in S_i, \quad \forall t \geq t^*.
\]

We note that given a trajectory consistent with a subdivision at time \( t^* \), for each set \( S_i \) and each \( t \geq t^* \), there exists a well-defined position \( z_{\alpha}(t) \) obtained by setting \( z_{\alpha}(t) = z_{\alpha}(t) \) for any \( \alpha \in S_i \). In the same spirit, if \( S_i = [\alpha(i, 1), \ldots, \alpha(i, n_i)] \) we can define
\[
\sigma_{S_i} = \sum_{j=1}^{n_i} \sigma_{\alpha(i,j)}, \\
w_{S_i}^j = w_{\alpha(i,1)}^j, \\
w_{S_i}^r = w_{\alpha(i,n_i)}^r,
\]
and if \( \tilde{Z} \) is also consistent, the discrepancy
\[
d_{S_i}(\tilde{Z}(t), Z(t)) = |\sigma_{S_i}| \cdot |z_{\alpha}(t) - z_{\alpha}(t)|.
\]

**Lemma 4.2.** For any \( k \in \{1, \ldots, s\} \) and any two sequences of numbers \( \{b_j\}_{j=k+1}^s \), \( \{a_i\}_{i=1}^k \) satisfying
\[
0 \leq b_j \leq 1, \quad a_i \geq 0 \quad \text{and} \quad \sum_{i=1}^k a_i = 1,
\]
then the matrix

\[
C = \begin{bmatrix}
1 & b_{k+1}a_1 & \ldots & b_sa_1 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
1 & b_{k+1}a_k & \ldots & b_sa_k \\
0 & 1-b_{k+1} & 0 & \cdot \\
0 & 0 & 1-b_s & \cdot 
\end{bmatrix}
\] (4.3)

is conservative.

The proof is a simple verification using Definition 3.10.

**Lemma 4.3.** Consider two trajectories \(Z\) and \(\tilde{Z}\) such that for all waves \(\alpha, \beta \in S\), and all \(t \geq t_*\),

\[
z_\alpha(t) = z_\beta(t) \quad \text{and} \quad \tilde{z}_\alpha(t) = \tilde{z}_\beta(t).
\]

Then there exists constant conservative matrices \(P\) and \(Q\) such that for all \(t \geq t_*\),

\[
d(\tilde{Z}(t), Z(t))_S = P \left[d_S(\tilde{Z}(t), Z(t))\right],
\]

\[
\left[d_S(\tilde{Z}(t), Z(t))\right] = Q \left[d(\tilde{Z}(t), Z(t))_S\right].
\] (4.4) (4.5)

**Proof.** Suppose that the waves in \(S\) are \(\alpha(1) < \cdots < \alpha(n)\). It suffices to take

\[
P = \begin{bmatrix}
\sigma_{\alpha(1)} \\
\vdots \\
\sigma_{\alpha(n)} \\
\sigma_S 
\end{bmatrix}
\quad \text{and} \quad Q = [1 \cdots 1].
\]

**Lemma 4.4.** Consider a subdivision of a consecutive set \(S \subset W\), say

\(S_1, \ldots, S_s\) (4.6)

and a pair of trajectories \(Z\) and \(\tilde{Z}\) for \(S\) such that for some \(t_*\) and \(T_* > t_*\), the following conditions are satisfied.

i) The trajectories \(Z(t)\) and \(\tilde{Z}(t)\) are consistent with the subdivision (4.6) for all \(t \in [t_*, T_*]\).

ii) \(Z(t) = \tilde{Z}(t)\) for all \(t \leq t_*\).

iii) \(\dot{Z}(t) = \tilde{Z}(t)\) for all \(t > T_*\).

iv) If \(Z(t) = [z_\alpha(t)]_{\alpha \in S}\) and \(\tilde{Z}(t) = [\tilde{z}_\alpha(t)]_{\alpha \in S}\), then for all \(t \in [t_*, T_*]\),

\[
\dot{z}_\alpha(t) = S(w_{S_1}, w_{S}) \quad \text{for all } \alpha \in S_1, \quad (4.7)
\]

\[
\dot{\tilde{z}}_\alpha(t) = S(w_{S_1}, w_{S}) \quad \text{for all } \alpha \in S. \quad (4.8)
\]
If \( k \) is the integer satisfying
\[
S(w^l_{S_k+1}, w^r_{S_k+1}) \leq S(w^l_S, w^r_S) \leq S(w^l_{S_k}, w^r_{S_k}),
\] (4.9)
then for all \( t \geq 0 \),
\[
\sum_{i=1}^k \left( \sum_{\alpha \in S_i} d_\alpha(\tilde{Z}(t), Z(t)) \right) = \sum_{i=k+1}^s \left( \sum_{\alpha \in S_i} d_\alpha(\tilde{Z}(t), Z(t)) \right).
\] (4.10)

**Proof.** We begin by noticing that condition \( i) \) allows us to introduce the notation (4.1) for the trajectories \( Z \) and \( \tilde{Z} \) when \( t \in [t_s, T_s] \). Moreover, the notation (4.1) will be further abbreviated to \( w^l_{i^*} = w^l_{S_{i^*}} \) and \( \sigma_i = \sigma_{S_i} \). For the moment, we assume that \( t \in [t_s, T_s] \) is fixed and that the trajectories \( Z \) and \( \tilde{Z} \) are indexed by the subdivision (4.6). In fact, it is immediate from condition \( ii) \) that the estimate (4.10) is trivial for \( t < t_s \). When \( t > T_s \), the condition \( iii) \) guarantees that for all \( \alpha \in S \),
\[
d_\alpha(\tilde{Z}(t), Z(t)) = d_\alpha(\tilde{Z}(T_s), Z(T_s)),
\]
and therefore that if we can prove (4.10) is true at time \( T_s \), then it will be true for all \( t > T_s \).

Without loss of generality, we may assume that \( t_s = 0 \). We now verify that it suffices to prove (4.10) for arbitrary translations \( \Delta_i \) of each of the positions \( z_i(t) \) and \( \tilde{z}_i(t) \). Writing \( \Delta = \left[ \Delta_i \right]_{i=1}^s \), for all \( t \) we have
\[
d_\alpha(\tilde{Z}(t), Z(t)) = |\sigma_i| \cdot |\tilde{z}_i(t) - z_i(t)|
\]
\[
= |\sigma_i| \cdot \left| (\tilde{z}_i(t) + \Delta_i) - (z_i(t) + \Delta_i) \right|
\]
\[
= d_\alpha(\tilde{Z}(t) + \Delta, Z(t) + \Delta).
\]

Now, pick a translation \( \Delta \) large enough that for each \( i = 0, \ldots, s \) and \( t \) belonging to the compact interval \( [0, T_s] \), we have
\[
z_i(t) + \Delta_i < z_{i+1}(t) + \Delta_{i+1}.
\]
Then for each \( \alpha \in S_i \), redefine the positions in \( Z(t) \) and \( \tilde{Z}(t) \) to be \( z_\alpha(t) + \Delta_i \) and \( \tilde{z}_\alpha(t) + \Delta_i \). Henceforth, we can assume that for \( t \in [0, T_s] \),
\[
z_i(t) < z_{i+1}(t).
\] (4.11)
Clearly, condition \( iv) \) then implies the same inequalities for \( \tilde{Z}(t) \).

Consider the function \( \tilde{\zeta} = \tilde{\zeta}(x, t) \), which is not a solution of the conservation law, and defined for \( t \in [0, T_s] \) as
\[
\tilde{\zeta}(x, t) = \begin{cases} 
  w^l_i & \text{if } x < \tilde{z}_i(t), \\
  w^r_i & \text{if } \tilde{z}_i(t) \leq x < \tilde{z}_{i+1}(t), \\
  w^r_s & \text{if } \tilde{z}_s(t) \leq x.
\end{cases}
\] (4.12)
By construction, the trajectories of the discontinuities in $\tilde{\zeta}$ are equal to $\tilde{Z}$ for $t \in [0, T_*]$. We will also need two entropy solutions of the conservation law. Let $\eta$ be defined using the initial data

$$
\eta(x, 0) = \begin{cases} 
    w_l & \text{if } x < z_*(0), \\
    w_r & \text{if } z_*(0) \leq x,
\end{cases}
$$

where the position $z_*(0)$ is chosen so that

$$
\int_{\mathbb{R}} \tilde{\zeta}(x, 0) - \eta(x, 0) \, dx = 0. \quad (4.13)
$$

For $t > 0$, let $z_*(t)$ be the position of the unique shock in $\eta$. The second entropy solution $\zeta = \zeta(x, t)$ is defined by its initial data $\zeta(x, 0) = \tilde{\zeta}(x, 0)$. By possibly increasing the size of $\Delta$, we can guarantee that none of the discontinuities in $\zeta$ will meet before time $T_*$. It is therefore clear that the trajectories of the discontinuities in $\zeta$ are equal to $Z(t)$ for $t \in [0, T_*]$.

The conservation law (1.1) and the identity (4.13) imply that for all $t$ we have

$$
0 = \int_{\mathbb{R}} \zeta(x, 0) - \eta(x, 0) \, dx = \int_{\mathbb{R}} \zeta(x, t) - \eta(x, t) \, dx. \quad (4.14)
$$

Notice that condition (4.8) implies that the discontinuity in $\eta$ moves at the same speed as the discontinuities in $\tilde{\zeta}$ and therefore, for all $t \in [0, T_*]$,

$$
0 = \int_{\mathbb{R}} \tilde{\zeta}(x, 0) - \eta(x, 0) \, dx = \int_{\mathbb{R}} \tilde{\zeta}(x, t) - \eta(x, t) \, dx. \quad (4.15)
$$

Subtracting (4.14) from (4.15), we find that for $t \in [0, T_*]$,

$$
0 = \int_{\mathbb{R}} \tilde{\zeta}(x, t) - \zeta(x, t) \, dx. \quad (4.16)
$$

Let $k$ be the integer satisfying (4.9). Then using (4.11), we obtain

$$
\bar{z}_i(t) \leq z_i(t), \quad \text{for } i \leq k, \text{ and}
$$

$$
z_i(t) \leq \bar{z}_i(t), \quad \text{for } k < i.
$$

This implies that for $x < z_k(t)$, $\tilde{\zeta}(x, t) \leq \zeta(x, t)$, while for $x \geq z_k(t)$ we have the opposite $\zeta(x, t) \leq \tilde{\zeta}(x, t)$. The identity (4.16) for $t \in [0, T_*]$ then becomes

$$
0 = \int_{-\infty}^{z_k(t)} \tilde{\zeta}(x, t) - \zeta(x, t) \, dx + \int_{z_k(t)}^{\infty} \tilde{\zeta}(x, t) - \zeta(x, t) \, dx
$$

$$
= -\sum_{i=1}^{k} |\sigma_i| \cdot |\bar{z}_i(t) - z_i(t)| + \sum_{i=k+1}^{s} |\sigma_i| \cdot |\bar{z}_i(t) - z_i(t)|
$$

$$
= -\sum_{i=1}^{k} d_i(\bar{Z}(t), Z(t)) + \sum_{i=k+1}^{s} d_i(Z(t), Z(t)).
$$

Condition iii) implies that this is sufficient to demonstrate the identity for all $t > T_*$.
Lemma 4.5. Consider a subdivision of a consecutive subset of \( S \subset W \), say
\[
S_1, \ldots, S_s
\]
and a pair of trajectories \( Z \) and \( \tilde{Z} \) for all waves \( W \) such that \( Z\vert_S \) and \( \tilde{Z}\vert_S \) satisfy the four conditions of Lemma (4.4) as well as
\begin{enumerate}
\item for \( t \geq t_* \), the waves in \( S \) belong to the same discontinuity in \( w \);
\item \( Z(t)\vert_{W\setminus S} = \tilde{Z}(t)\vert_{W\setminus S} \quad \forall t > 0 \);
\item \( \dot{Z}(t) = \dot{\tilde{Z}}(t) = X(t) \) for \( t > T_* \);
\item for all \( \alpha \in S_i \), all \( \beta \in S_{i+1} \) and all \( t \in [t_*, T_*] \),
\[
z_\alpha(t) \leq z_{\beta}(t).
\]
\end{enumerate}

Under these conditions, there exists a one-parameter family of conservative matrices \( S(t) \) such that for all \( t \),
\[
d(Z(t), X(t)) \leq S(t) \cdot d(\tilde{Z}(t), X(t)).
\]

Proof. The notation can be simplified a bit by observing that condition \( i) \) allows us to use the notation (4.1). We will go a bit further and simply write \( w_{S^i}^l = w_{S^i}^r \) and \( d_s(\cdot, \cdot) = d_{S^i}(\cdot, \cdot) \). Notice also that condition \( v) \) implies that there exists a position \( x(t) \) such that for all \( \alpha \in S \) and all \( t \geq t_* \), \( x(t) = x_\alpha(t) \). Because of condition \( vi) \), it suffices to demonstrate the bound (4.19) only for the waves in \( S \). Finally, we remark that for all waves \( \alpha \in S \), the conditions \( v) \) and \( viii) \) imply that for all \( t \geq t_* \)
\[
z_i(t) \leq z_{i+1}(t),
\]
\[
\tilde{z}_i(t) \leq \tilde{z}_{i+1}(t).
\]

The main difficulty in this proof is dealing with the fact that the position \( x(t) \) of the discontinuity \( S \) in \( w \) is arbitrary. Let \( k \) be the integer guaranteed by Lemma 4.4 and appearing in identity (4.10). At a fixed time \( t \geq t_* \), one of three cases can occur. Either
\begin{enumerate}
\item \( x(t) \leq \tilde{z}_1(t) \leq \cdots \leq \tilde{z}_s(t) \),
\item \( \tilde{z}_1(t) \leq \cdots \leq \tilde{z}_a(t) \leq x(t) \), or
\item there exists \( j \) such that \( \tilde{z}_j(t) \leq x(t) \leq \tilde{z}_{j+1}(t) \).
\end{enumerate}

Consider the first case. We will omit the time dependence of the trajectories since the time \( t \) will be fixed throughout. From the construction of \( k \) and condition \( vii) \), it is known that for \( i \leq k \), \( \tilde{z}_i(t) \leq \tilde{z}_i(t) \) for all \( t \geq t_* \). Therefore using condition
 mechanisms for error propagation and cancellation

\[ ii), \; x(t) \leq \bar{z}_i(t) \leq z_i(t) \text{ for } t \geq t_*, \text{ and} \]
\[ d_i(X, Z) = d_i(X, \bar{Z}) + d_i(\bar{Z}, Z) \]
\[ = d_i(X, \bar{Z}) + d_i(\bar{Z}, Z) \cdot \frac{\sum_{j=k+1}^s d_j(\bar{Z}, Z)}{\sum_{j=k+1}^s d_j(\bar{Z}, Z)} \]
\[ = d_i(X, \bar{Z}) + \sum_{j=k+1}^s d_i(\bar{Z}, Z) \cdot \frac{d_j(X, \bar{Z})}{\sum_{j=k+1}^s d_j(\bar{Z}, Z)} \cdot \frac{d_j(\bar{Z}, Z)}{\sum_{j=k+1}^s d_j(\bar{Z}, Z)} \]
\[ = d_i(X, \bar{Z}) + \sum_{j=k+1}^s \left( \frac{d_j(\bar{Z}, Z)}{d_j(X, \bar{Z})} \cdot \frac{d_j(\bar{Z}, Z)}{\sum_{j=k+1}^s d_j(\bar{Z}, Z)} \right) d_j(X, \bar{Z}). \quad (4.21) \]

From condition \( \text{viii}) \), we find that for all \( i > k \), we have \( z_i(t) \leq z_i(t) \) and therefore, with the help of condition \( \text{iv}) \), \( x(t) \leq \bar{z}_k(t) \leq \bar{z}_k(t) \leq z_i(t) \). Using only \( x(t) \leq z_i(t) \leq \bar{z}_i(t) \) for \( i > k \), we find
\[ d_i(X, Z) = d_i(X, \bar{Z}) - d_i(\bar{Z}, Z) = d_i(X, \bar{Z}) \left( 1 - \frac{d_i(\bar{Z}, Z)}{d_i(X, \bar{Z})} \right). \quad (4.22) \]

Define the vectors
\[ D \triangleq \begin{bmatrix} d_1(Z, X) \\ \vdots \\ d_s(Z, X) \end{bmatrix}, \quad \bar{D} \triangleq \begin{bmatrix} d_1(\bar{Z}, X) \\ \vdots \\ d_s(\bar{Z}, X) \end{bmatrix}. \quad (4.23) \]

Identities \( (4.21) \) and \( (4.22) \) show that \( D \leq C(t)\bar{D} \) if the conservative matrix \( C(t) \) is defined according to \( (4.3) \) with
\[ a_i = \frac{d_i(\bar{Z}, Z)}{\sum_{i=k+1}^s d_i(\bar{Z}, Z)}, \quad \text{for } i = 1, \ldots, k, \quad (4.24) \]
\[ b_j = \frac{d_j(\bar{Z}, Z)}{d_j(X, \bar{Z})}, \quad \text{for } j = k + 1, \ldots, s. \quad (4.25) \]

The matrix \( C(t) \) is conservative because the parameters \( a_i \) and \( b_j \) satisfy the conditions of Lemma 4.2, most importantly, because of Lemma 4.4,
\[ \sum_{i=1}^k a_i = \frac{\sum_{i=1}^k d_i(\bar{Z}, Z)}{\sum_{i=k+1}^s d_i(\bar{Z}, Z)} = 1. \]

We now prove a restricted estimate for \( d(Z, X)|_S \) and \( d(\bar{Z}, X)|_S \). For each set \( S_i \), construct the matrices \( P_i \) and \( Q_i \) given by Lemma 4.3 and define
\[ P \triangleq \begin{bmatrix} P_1 \\ \vdots \\ P_s \end{bmatrix}, \quad Q \triangleq \begin{bmatrix} Q_1 & \cdots & Q_s \end{bmatrix}. \quad (4.26) \]

It then follows that
\[ d(\bar{Z}, X)|_S = P\bar{D} \leq PC(t)\bar{D} = PC(t)Qd(\bar{Z}, X)|_S. \quad (4.27) \]
To prove the result (4.19) for all waves in $\mathcal{S}$, take $S(t) = PC(t)Q$.

Consider the third case where $z_j(t) \leq x(t) \leq \tilde{z}_{j+1}(t)$, for some fixed $t \geq t_\star$. The simplest situation occurs when $j = k$ and we also have the stronger bounds $z_k(t) \leq x(t) \leq z_{k+1}(t)$. Under these conditions, and with the help of the definition of $k$, we conclude that for $i \leq k$

$$z_i(t) \leq z_i(t) \leq x(t),$$

(4.28)

while for $i > k$,

$$x(t) \leq z_i(t) \leq \tilde{z}_i(t).$$

(4.29)

Therefore, for all $i$,

$$|\sigma_{S_i} \cdot |z_i(t) - x(t)| \leq |\sigma_{S_i} \cdot |\tilde{z}_i(t) - x(t)|,$$

(4.30)

hence proving (4.19) by simply taking $C(t)$ equal to the identity matrix.

Consider the third case with the additional restriction that $x(t) \leq z_k(t)$. The proof of (4.19) when $z_{k+1}(t) \leq x(t)$ is similar and omitted. For any $i \in \{1, \ldots, k\}$, the triangle inequality guarantees

$$d_i(Z, X) \leq d_i(X, \tilde{Z}) + d_i(\tilde{Z}, Z).$$

and therefore (4.21) holds as an inequality. For $i \in \{k+1, \ldots, s\}$, we always have $x(t) \leq z_k(t) \leq z_i(t) \leq \tilde{z}_i(t)$ and therefore (4.22) holds as an equality. Taken together, they show that the matrix of Lemma 4.2 defined according to (4.24) and (4.25) still implies the result (4.19) for all $t \geq t_\star$.

This completes the proof of Lemma 4.5.

Lemma 3.11 is a consequence of the previous estimate.

**Proof.** [Lemma 3.11] Recall the definitions of subdivisions (3.10)-(3.12) as well as of the trajectories (3.14) and (3.15). We will build a sequence of intermediate trajectories relating $X^{(1)}$ to $X^{(2)}$ by producing one correction for each set $\mathcal{F}^{(i)}$ in the subdivision (3.10).

Define the trajectory $Z^{(i)}(t) = [z^{(i)}(t)]_\alpha$ to satisfy

$$z^{(i)}(t) = \begin{cases} 
  x^{(2)}(t) & \text{if } \alpha \in \mathcal{F}^{(1)} \wedge \cdots \wedge \mathcal{F}^{(i-1)}, \\
  x^{(1)}(t) & \text{if } \alpha \in \mathcal{F}^{(i)} \wedge \cdots \wedge \mathcal{F}^{(m)}. 
\end{cases}$$

(4.31)

It is sufficient to prove that for each $i \in \{1, \ldots, m\}$ there exists a one-parameter family of conservative matrices $B^{(i)}(t)$ such that

$$d(Z^{(i+1)}(t), X(t)) \leq B^{(i)}(t) \cdot d(Z^{(i)}(t), X(t)), \quad \forall t.$$  

(4.32)

Inequality (3.17) will follow from this estimate by observing that $X^{(1)} = Z^{(1)}$ and $X^{(2)} = Z^{(m+1)}$ and then applying it successively from $i = 1$ to $m$.

We will use Lemma 4.5 to prove inequality (4.32). In the notation of Lemma 4.5, set $\mathcal{S} = \mathcal{F}^{(i)}$, $Z = Z^{(i+1)}$ and $\tilde{Z} = Z^{(i)}$. As subdivision (4.17) of $\mathcal{S}$, take the
subdivision (3.11) of $F^{(i)}$. With these definitions, conditions i), ii), iii) (stated in Lemma 4.4), as well as vii), are satisfied. To verify condition iv), we notice that for $t \in [t_*, T_*]$

$$Z(t)|_S = X^{(2)}(t)|_{F^{(i)}} = F^{(i)}(t),$$

which is the free trajectory of $F^{(i)}$, see Definition 3.8. Since the subdivision of $F^{(i)}$ corresponds to distinct discontinuities in the entropy solution $V$ defining the free trajectory, we have that for $t \in [t_*, T_*]$ and each $\alpha \in S_i$,

$$\dot{z}_\alpha(t) = S(w^l_{S_\alpha}, w^r_{S_\alpha}).$$

(4.34)

This shows (4.7) of condition iv). On the other hand, $\tilde{Z}(t)|_S = X^{(1)}|_{F^{(i)}}$ is defined by (3.14), which, because $F^{(i)}$ represents a single discontinuity in $w$ when $t \geq t_*$, travels at a constant speed

$$\dot{\tilde{z}}_\alpha(t) = S(w^l_{S_\alpha}, w^r_{S_\alpha}), \quad \forall \alpha \in S.$$

(4.35)

This verifies (4.8) of condition iv). Condition v) of Lemma 4.5 is satisfied with $S = F^{(i)}$. To check vii), it suffices to examine the definition of $Z^{(i)}$. Definitions (3.14) and (3.15) imply that condition vii) is satisfied for $Z$ and $\tilde{Z}$. Since the positions $z_\alpha$ are those of the discontinuities in an exact solution, then condition viii) must also be satisfied for $t \in [t_*, T_*]$, that is for $t$ less than the time of the next interaction in the free trajectories. At later times, $\dot{z}_\alpha(t) = \dot{z}_\beta(t)$ for all $\alpha, \beta \in S$ and since condition viii) is satisfied at time $T_*$, then it must continue to hold at all later times.

All the conditions of Lemma 4.5 all satisfied and therefore there exists $S(t) = B^{(i)}(t)$ such that (4.32) holds. This is completes the proof of Lemma 3.11.

The next result is to Lemma 3.12 as Lemma 4.5 is to Lemma 3.11. It provides the basic construction that must be repeated inductively in the proof of Lemma 3.12.

**Lemma 4.6.** Consider a subdivision of a consecutive subset of $S \subset W$, say

$$S_1, \ldots, S_s$$

(4.36)

and a pair of trajectories $Z$ and $\tilde{Z}$ such that $Z|_S$ and $\tilde{Z}|_S$ satisfy the four conditions of Lemma 4.4. Let the ordered set of waves $S_i$ be written $[\alpha(i, 1), \ldots, \alpha(i, n_i)]$. Further assume that

v) for all $t \geq t_*$ and with the integer $k$ of Lemma 4.4, we have

$$z_{\alpha(s,n_s)}(t) \leq \cdots \leq z_{\alpha(1,1)}(t),$$

(4.37)

$$\tilde{z}_{\alpha(s,n_s)}(t) \leq \cdots \leq \tilde{z}_{\alpha(1,1)}(t),$$

(4.38)

$$z_{\alpha(i,j)}(t) \leq \tilde{z}_{\alpha(i,j)}(t), \quad \text{if } i > k,$$

(4.39)

$$\tilde{z}_{\alpha(i,j)}(t) \leq z_{\alpha(i,j)}(t), \quad \text{if } i \leq k;$$
Proof. The arguments used in this proof are essentially the same as those first seen in Lemma 4.5. Unfortunately the position of the waves $X|_S$ relative to $Z$ and $\tilde{Z}$ are more complex and require the analysis of several cases. We begin the proof by observing that conditions \textit{iii)} and \textit{vi)} imply that (4.41) is trivial for waves in $W \setminus S$. Without further mention, we will assume that the indices of trajectories $X$, $Z$, and $\tilde{Z}$ have been restricted to $S$.

Fix a time $t > t_*$ throughout the remainder and omit the time-dependence of the trajectories. The properties of Glimm’s scheme provide an approximate solution whose waves have retained their initial order

$$x_{\alpha(1,1)} \leq \cdots \leq x_{\alpha(1,n)} \leq x_{\alpha(2,1)} \leq \cdots \leq x_{\alpha(s,1)} \leq \cdots \leq x_{\alpha(s,n_s)}.$$  \hspace{1cm} (4.42)

Condition \textit{v)} provides bounds for the waves in $Z$ and $\tilde{Z}$ that go in the opposite order for $t > t_*$, as seen in (4.37) and (4.38).

We identify three different cases, similar to those seen in Lemma 4.5. Either

1. $\tilde{z}_{\alpha(s,n_s)} \leq \cdots \leq \tilde{z}_{\alpha(1,1)} \leq x_{\alpha(1,1)} \leq \cdots \leq x_{\alpha(s,n_s)}$;
2. $x_{\alpha(1,1)} \leq \cdots \leq x_{\alpha(s,n_s)} \leq \tilde{z}_{\alpha(s,n_s)} \leq \cdots \leq \tilde{z}_{\alpha(1,1)}$;
3. or there exists some $\alpha(m,n) \in S$ such that

$$x_{\alpha(m,n)} \leq \tilde{z}_{\alpha(m,n)} \quad \text{and} \quad \tilde{z}_{\alpha(m,n+1)} \leq x_{\alpha(m,n+1)}.$$ \hspace{1cm} (4.43)

The fact that the waves in $\tilde{Z}$ and $X$ appear in the opposite order guarantees the existence of the wave $\alpha(m,n)$. We also observe that in the third case, it might be necessary to interpret $\alpha(m,n+1)$ as $\alpha(m+1,1)$ if $n = n_m$.

Consider the first case. The analysis of the second case is similar and will therefore be omitted. Pick any wave $\alpha(i,j)$ with $i \leq k$, $k$ satisfying (4.9) of Lemma 4.4, and observe that (4.39) implies either $\tilde{z}_{\alpha(i,j)} \leq z_{\alpha(i,j)} \leq x_{\alpha(i,j)}$ or $z_{\alpha(i,j)} \leq x_{\alpha(i,j)} \leq z_{\alpha(i,j)}$. For both sets of inequalities

$$d_{\alpha(i,j)}(X, \tilde{Z}) \leq d_{\alpha(i,j)}(X, Z) + d_{\alpha(i,j)}(Z, \tilde{Z}).$$

Repeating the calculation done in Lemma 4.5 for formula (4.21), we can obtain

$$d_{\alpha(i,j)}(X, \tilde{Z}) \leq d_{\alpha(i,j)}(X, Z)$$

$$+ \sum_{p=k+1}^{n_p} \sum_{q=1}^{n_p} \left( \frac{d_{\alpha(p,q)}(Z, \tilde{Z})}{d_{\alpha(p,q)}(X, Z)} \cdot \frac{d_{\alpha(i,j)}(Z, \tilde{Z})}{\sum_{e=k+1}^{n_p} \sum_{f=1}^{n_p} d_{\alpha(e,f)}(Z, \tilde{Z})} \right) d_{\alpha(i,j)}(X, Z).$$
Mechanisms for error propagation and cancellation

When the wave \( \alpha(i,j) \) is chosen with \( i > k \), then \( z_{\alpha(i,j)} \leq \zeta_{\alpha(i,j)} \leq x_{\alpha(i,j)} \), which provides the identity

\[
d_{\alpha(i,j)}(X, \tilde{Z}) = d_{\alpha(i,j)}(X, Z) - d_{\alpha(i,j)}(Z, \tilde{Z}) = d_{\alpha(i,j)}(X, Z) \left( 1 - \frac{d_{\alpha(i,j)}(Z, \tilde{Z})}{d_{\alpha(i,j)}(X, Z)} \right).
\]

These two identities imply that there exists a conservative matrix \( C(t) \) of the form (4.3), with

\[
b(i,j) = \frac{d_{\alpha(i,j)}(Z, \tilde{Z})}{d_{\alpha(i,j)}(X, Z)}, \quad \text{for } i = k + 1, \ldots, s, j = 1, \ldots, n_i, \quad (4.44)
\]

\[
a(i,j) = \frac{d_{\alpha(i,j)}(Z, \tilde{Z})}{\sum_{e=k+1}^{s} \sum_{f=1}^{n_e} d_{\alpha(e,f)}(Z, Z)}, \quad \text{for } i = 1, \ldots, k, j = 1, \ldots, n_i, \quad (4.45)
\]

such that (4.41) is satisfied for the waves in \( S \). The conservation property of \( S(t) \) is equivalent to the main identity of Lemma 4.4,

\[
\sum_{i=1}^{k} \sum_{j=1}^{n_i} a(i,j) = \frac{\sum_{i=1}^{k} \sum_{j=1}^{n_i} d_{\alpha(i,j)}(Z, \tilde{Z})}{\sum_{e=k+1}^{s} \sum_{f=1}^{n_e} d_{\alpha(e,f)}(Z, Z)} = 1.
\]

Consider now the third case. Suppose that \( m \leq k \), although \( m > k \) could be treated in a similar manner. If a wave \( \alpha(i,j) \in S \) satisfies \( \alpha(i,j) \leq \alpha(k, n_k) \), the triangle inequality shows that

\[
d_{\alpha(i,j)}(X, \tilde{Z}) \leq d_{\alpha(i,j)}(X, Z) + d_{\alpha(i,j)}(Z, \tilde{Z}).
\]

If the wave \( \alpha(i,j) \) satisfies \( \alpha(m,n) < \alpha(k+1,1) \leq \alpha(i,j) \), then the application (4.39), (4.38), (4.43), (4.42) leads to

\[
z_{\alpha(i,j)} \leq \zeta_{\alpha(i,j)} \leq \zeta_{\alpha(m,n+1)} \leq x_{\alpha(m,n+1)} \leq x_{\alpha(i,j)},
\]

\[
d_{\alpha(i,j)}(X, \tilde{Z}) = d_{\alpha(i,j)}(X, Z) - d_{\alpha(i,j)}(Z, \tilde{Z}).
\]

The matrix (4.3) defined by (4.44) and (4.45) now guarantees that the components of \( S(t)d(X, Z) \) (restricted to \( S \)) are

\[
d_{\alpha(i,j)}(X, Z) + d_{\alpha(i,j)}(Z, \tilde{Z}) \quad \text{for } i = 1, \ldots, k, j = k, \ldots, n_i, \text{ and}
\]

\[
d_{\alpha(i,j)}(X, Z) - d_{\alpha(i,j)}(Z, \tilde{Z}) \quad \text{for } i = k + 1, \ldots, s, j = 1, \ldots, n_i.
\]

For all waves \( \alpha(i,j) \), these components bound \( d_{\alpha(i,j)}(X, \tilde{Z}) \). This completes the proof of Lemma 4.6.

The proof of the second important tool of Section 3 can now be presented. To make the proof a little clearer, it will be convenient to use the following definition.

**Definition 4.7.** If \( F(t) \) are the free trajectories of a consecutive set of waves \( \mathcal{F} \), then we say that \( S \subset \mathcal{F} \) is a discontinuity in \( F(t) \) at time \( t \) if the waves \( S \) form a discontinuity in the entropy solution \( V(\cdot, t) \) of Definition 3.8.
Proof. [Lemma 3.12] Without loss of generality, we may assume that $\mathcal{G} = \mathcal{W}$ since in any case, the objective is to demonstrate an estimate that involves only waves in $\mathcal{G}$. For both $U$ and $\tilde{U}$, we assume that interactions in the free trajectories occur only between pairs of discontinuities. This is true for generic initial data because the flux $f$ in (1.1) is strictly convex.

Suppose that interactions between discontinuities inside the free trajectories $G(t), G^{(1)}(t), \ldots, G^{(k)}(t)$, for $t \in [0, T]$ occur at the $N$ times

$$0 = t_0^{(0)} < t_1^{(1)} < t_2^{(2)} < \cdots < t_N^{(N)} \leq T,$$

and define $t_0^{(N+1)} = T$.

Define inductively the trajectories $Z^{(i)}$ according to

$$Z^{(i)}(t) = \tilde{U}(t) \quad \text{for } t \leq t_i^{(i)},$$

$$\dot{Z}^{(i)}(t) = \tilde{U}(t) \quad \text{for } t > t_i^{(i)}.$$ (4.46)

With this definition and the identity $\tilde{U}(0) = U(0)$, we have $Z^{(N+1)}(t) \equiv \tilde{U}(t)$ and $Z^{(0)}(t) \equiv U(t)$ for all $t$. We will show that for each $i = 0, \ldots, N$, there exists a conservative matrix $C^{(i)}(t)$ satisfying

$$d(Z^{(i+1)}(t), X(t)) \leq C^{(i)}(t)d(Z^{(i)}(t), X(t)), \quad \forall t.$$ (4.47)

Estimate (3.19) will follow from the repeated application of this inequality.

We now make the crucial observation that, during the time interval $[0, T]$ shock interactions in $\tilde{U}$ must always occur before those in $U$. In practice, this translates into the fact that if two waves $\alpha$ and $\beta$ have the same position in $\tilde{U}(t)$, then they must also have the same position in $U(t)$ (although $u_\alpha(t)$ is not necessarily equal to $u_\beta(t)$). For $t \in [0, T]$, the free trajectories $G^{(i)}(t) \equiv U(t)|_{\mathcal{G}^{(i)}}$ are the positions of waves in some entropy solution, say $V^{(i)}$, defined using only the waves $\mathcal{G}^{(i)}$ in the initial data; see Definition 3.8. The free trajectories $G(t) \equiv U(t)$ are the positions of waves in an entropy solution $V$ defined using all the waves $\mathcal{G}$ in the initial data. The additional waves, and hence discontinuities, in $V$ can only make the discontinuities generated by $\mathcal{G}^{(i)}$ meet at a possibly earlier time. In this sense, if the positions of waves in $V^{(i)}$ are equal at time $t$, and hence the discontinuities from which they originated must have interacted, then these discontinuities must have also interacted in $V$ before time $t$.

Returning to the proof of (4.47), we will another sequence of intermediate trajectories connecting $Z^{(i)}$ to $Z^{(i+1)}$. Clearly, $Z^{(i)}(t_i^{(i)}) = Z^{(i+1)}(t_i^{(i)})$, and $\dot{Z}^{(i)}(t) = \dot{Z}^{(i+1)}(t)$ for $t > t_i^{(i+1)}$. The definition of the times $t_i^{(j)}$ implies that during the time interval $[t_i^{(i)}, t_i^{(i+1)}]$, the speeds of the trajectories $U$ and $\tilde{U}$ are constant. Let the discontinuities in $Z^{(i)}(t_i^{(i)}) \equiv U(t_i^{(i)})$ be represented by the consecutive sets of waves $p_1, p_2, \ldots, p_p$.

We will now define a sequence of intermediate trajectories relating $Z^{(i)}$ to $Z^{(i+1)}$. 

Mechanisms for error propagation and cancellation

Set \(Y^{(i,j)} = [y^{(i,j)}(t)]_{\alpha \in \mathcal{W}}\) to be trajectories

\[
y^{(i,j)}(t) = \begin{cases} 
  z^{(i+1)}(t) & \text{if } \alpha \in \mathcal{P}_1 \land \ldots \land \mathcal{P}_{j-1} \text{ when } t \leq t^{(i+1)}_s, \\
  z^{(i)}(t) & \text{if } \alpha \in \mathcal{P}_j \land \ldots \land \mathcal{P}_p \\
  \dot{Z}^{(i)}(t) & \text{for } t > t^{(i+1)}_s
\end{cases}
\]  

(4.48)

According to this definition, \(Y^{(i,1)} = Z^{(i)}\) and \(Y^{(i,p+1)} = Z^{(i+1)}\). We will demonstrate (4.47) by constructing conservative matrices \(B^{(i,j)}(t)\) such that for \(j = 1, \ldots, p\) and all \(t < T\),

\[
d(Y^{(i,j+1)}(t), X(t)) \leq B^{(i,j)}(t) \cdot d(Y^{(i,j)}(t), X(t)).
\]  

(4.49)

As before, for each \(i\), we will deduce (4.47) by using composition of this inequality.

In order to demonstrate the existence of \(B^{(i,j)}\) in (4.49), we will apply Lemma 4.6 using \(\bar{Z} = Y^{(i,j+1)}\), \(Z = Y^{(i,j)}\), \(t_s = t^{(i)}_s\), \(T_s = t^{(i+1)}_s\), and \(\mathcal{S} = \mathcal{P}_j\). The earlier observation we made concerning the precedence of the interactions in \(\bar{U}\) over \(U\) implies that each discontinuity \(\mathcal{P}_j\) in \(\bar{U}(t^{(i,j)}_s) = Z^{(i)}(t^{(i,j)}_s)\) appears as the union of one or more discontinuities in \(U(t^{(i,j)}_s)\). This property therefore induces a decomposition of \(\mathcal{S} = \mathcal{P}_j\) into \(s\) waves \(\mathcal{S}_1, \ldots, \mathcal{S}_s\) which we will use as the subdivision (4.36) of Lemma 4.6.

These objects clearly satisfy conditions \(i\), \(ii\), \(iii\), \(vi\), and \(vii\). During the time interval \([t^{(i,j)}_s, t^{(i,j+1)}_s]\), the waves in \(Y^{(i,j)}(t)\) move at the speed of the waves in \(U\), according to definition (4.48), while the waves in \(Y^{(i,j+1)}(t)\) move at the speed of \(\bar{U}\). Since each set \(\mathcal{S}_i\) corresponds to a discontinuity in \(U\), it is clear that for \(t \in [t_s, T_s]\), and for each \(\alpha \in \mathcal{S}_i\),

\[
\dot{z}_\alpha(t) = y^{(i,j)}(t) = S(w^l_{\mathcal{S}_i}, w^r_{\mathcal{S}_i}),
\]  

(5.50)

while for all \(\alpha \in \mathcal{S}\),

\[
\dot{z}_\alpha(t) = y^{(i,j+1)}(t) = S(w^l_{\mathcal{S}}, w^r_{\mathcal{S}}).
\]  

(5.51)

It remains only to show that condition \(v\) of Lemma 4.6 is satisfied. Because of (4.9) and condition \(vii\), the position of all the waves in \(\mathcal{P}_j\) are equal in \(Y^{(i,j+1)}(t)\) during the time interval \([t^{(i,j)}_s, t^{(i,j+1)}_s]\). Using the definition of the integer \(k\) in (4.9), at time \(t = t^{(i,j+1)}_s\), we find

\[
\begin{align*}
  z^{(s,n_j)}(t) & \leq \ldots \leq z^{(k+1,1)}(t) \\
  \leq z^{(s,n_j)}(t) & = \ldots = z^{(1,1)}(t) \\
  \leq z^{(k,n_k)}(t) & \leq \ldots \leq z^{(1,1)}(t).
\end{align*}
\]

The second condition of (4.48) coupled with the inequalities above only guarantee that for all \(t \geq t^{(i+1)}_s\)

\[
\begin{align*}
  z^{(i,j)}(t) & \leq \bar{z}^{(i,j)}(t) & \text{for } i \leq k, \\
  \bar{z}^{(i,j)}(t) & \leq z^{(i,j)}(t) & \text{for } i > k.
\end{align*}
\]
A more careful analysis of the wave interactions in $G$ and $G^{(i)}$ is required to show that (4.37) and (4.38) continue to hold for $t \geq T_\epsilon = T^{(i+1)}_\epsilon$. It suffices to show that for all $t \geq T_\epsilon$

$$\bar{z}_{\alpha(s,n,i)}(t) \leq \cdots \leq \bar{z}_{\alpha(1,1)}(t).$$

(4.52)

Identical inequalities will then hold for $\bar{Z}$ because of (4.48).

We begin by demonstrating (4.52) for $t \in [T_\epsilon, T]$. During that time interval, the wave speeds in $\bar{Z}(t) = \tilde{Z}(t) = \tilde{U}(t)$ change only at a finite number of times. We will show that if (4.52) holds just before such an interaction time $\tilde{t}$, then it must hold immediately afterwards, thus proving the inequalities by induction. If the speed of a wave $\alpha \in S \cap G^{(i)}$ changes, then it must be because an interaction occurred in the free trajectories $G^{(i)}(t) = \tilde{U}(t)|_{G^{(i)}}$. Suppose that after the interaction at time $t$, two waves $\alpha < \beta \in G^{(i)}$ possess the same speed (the case $\beta < \alpha$ is similar). Since the positions of these waves are obtained from an entropy solution, the inequality $\alpha < \beta$ implies that

$$\dot{g}_\beta^{(i)}(\tilde{t}^-) \leq \dot{g}_\beta^{(i)}(\tilde{t}^+) \leq \dot{g}_\alpha^{(i)}(\tilde{t}^-) \leq \dot{g}_\alpha^{(i)}(\tilde{t}^+).$$

If $\gamma \in S$ is a wave whose speed didn’t change and, say, $\gamma < \alpha$ then these inequalities, the identity $z_\alpha(t) = g_\alpha^{(i)}(t)$ and (4.52) at time $\tilde{t}^-$ show

$$z_\alpha(\tilde{t}^+) \leq z_\alpha(\tilde{t}^-) \leq z_\gamma(\tilde{t}^-) = z_\gamma(\tilde{t}^+).$$

A similar inequality holds if $\beta < \gamma$, which shows (4.52) for times immediately following $\tilde{t}$ and thus, by induction, for all $t \in [T_\epsilon, T]$.

We now show that (4.52) is true for all $t \geq T$. After that time, the definitions of $U$ and $\tilde{U}$ state that $\dot{U}(t) = \tilde{U}(t) = \dot{X}(t)$, and the definitions (4.46) and (4.48) provide $\dot{U}(t) = \dot{Z}(i)(t) = \dot{Y}^{(i,j)}(t)$. The inequalities (4.52) then hold because for any two waves $\alpha < \beta$ in $W$, we have $x_\alpha(t) \leq x_\beta(t)$ and therefore, because the approximate solution $w$ is decreasing

$$\dot{x}_\alpha(t) = S(w_\alpha^-(t), w_\alpha^+(t)) \geq \dot{x}_\beta(t) = S(w_\beta^-(t), w_\beta^+(t)).$$

Lemma 4.6 then provides matrices $B^{(i,j)}(t)$, thus proving (4.49) and, by composition, estimate (4.47). This completes the proof of Lemma 3.11.

5. Conclusions and Future Work

This work is, as far as we can tell, a novel approach to the $L^1$ stability of conservation laws. The technique requires that the approximate solution be described by some form of the method of characteristics. On the other hand, it does not require explicit knowledge of the solution.

Numerical results [10] and unpublished mathematical results indicate that Theorem 3.1 probably continues to hold for arbitrary initial data. First, for an appropriate definition of the discrepancy $d_\beta(\cdot, \cdot)$ for a rarefaction $\beta$, Lemma 3.4 should continue to hold. If at time $t$, the rarefaction has yet to interact in either $u$ or $w$,
then we can show $d_\beta(Y(t_n), X(t_n)) = |\sum_k R(\beta, t_k)|$ which is sufficient. If a rarefaction $\beta$ interacts with a shock $\alpha$ in either $u$ or $w$, then partial results suggest that there exists transformations of either $X^{(2)}$ or $X^{(0)}$ and appropriate conservative matrices for which we continue to have bounds like those described in Lemmas 4.5 and 4.6. This should permit the waves $\alpha$ and $\beta$ to be effectively removed from the remainder of the analysis and thus reduce the problem to the case of pure shock data, as treated in this paper. The abstract formulations of Lemmas 4.5 and 4.6 indicate that these tools continue to hold, even in the presence of rarefactions. This is a topic of ongoing research.

This partial result also leads us to ask a few questions. The error estimate suggests that one not use random sequences $\{\theta_k\}$ but rather sequences for which the error estimators are smallest. This could lead to an interesting form of Glimm’s scheme where adaptivity is done with respect to $\theta_k$. The current approach is especially interesting because it offers the possibility of demonstrating stability without the use of a global functional. For example, this property could be useful to treat stability and existence for solutions to nonlinear conservation laws with only $L^\infty(\mathbb{R})$ initial data. The techniques appear to be compatible with the techniques of Glimm and Lax [4] and might lead to a measurement of the rate of convergence in $L^1$ of the entropy solution towards the asymptotic N-wave solution for $2 \times 2$ systems. In this respect, we also add that the finiteness of the number of waves, essential to our inductive proof, still holds for $2 \times 2$ systems but not for general $n \times n$ systems.

Finally, we note that this work provides the first explicit example of error cancellation for a finite difference scheme, outside of the adjoint framework [13]. It also answers a question raised by Risebro [14].

Acknowledgments

I thank James Glimm for his early interest in this work. Funding from the National Science and Engineering Council of Canada was also greatly appreciated.

References


